TA C: Mathematics and mathematics education

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Introduction

The guidelines as outlined in the announcement of the congress could be described as follows.

- Trends in the mathematical sciences and their influence on mathematics education
- The role of research mathematicians in mathematics education
- New and old mathematical topics, and the balances between them, in mathematics curricula
- The mathematics educator: Mathematician or pedagogue?

In the opening remarks of the afternoon *Jean-Pierre Bourguignon* gave his vision of the many issues to be discussed. It is a fact that mathematics is part of the school curricula at schools in almost every country in the world at various stages of the education process. The issue to concentrate on is whether and, if so, how the mathematical training in schools should be influenced by the evolution of mathematics as a science and in its relation to society. It is widely accepted that exposure to mathematical ideas at school is part of the education to systematic thinking, and basic mathematical objects such as numbers and geometrical figures are used for that purpose. The following questions arise. What about giving a glimpse of both the achievements of present day mathematics and of its multiple uses in society? Should one make the fact perceptible that mathematics as a science is thriving and is presently developing at an unprecedented pace?
First, an overview of the present situation would be useful. It is certainly desirable to form a global idea of the content of the pre- and in-service training of teachers. Of particular importance is the impact of this training on the personal relation teachers entertain with mathematics.

How can teachers maintain contact with present-day mathematics and the new involvement of mathematical facts, products and ideas in many areas of the society? Which documents are available for that purpose? By whom and how can their requests for contacts and explanations be answered? What kinds of contacts with research mathematicians are institutionally organized:

– Conferences and workshops?
– Cooperative projects in schools?
– Internships in research labs?
– Other modes of exchange?

It would be most important to attempt to analyze difficulties or insufficiencies that can be identified:

– At the level of training
– In the contacts with present-day mathematics and mathematicians.

Looking towards the future some avenues have to be explored:
- Are there web resources that can contribute? If yes, in what format?
- What kind of events, or structures, can help meet the needs?
- Who should be responsible for establishing and maintaining them, teachers or research mathematicians?
- What kind of agencies should take the lead in such matters?
- How can one get users of mathematics to testify about their uses?

After the introduction the following presentations based on previously distributed papers were given. We shall give a brief outline of them in the sequel.

Presentations

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The long way (from primary school to the end of secondary school) for constructing the concept of limit

The concepts of limit, continuity, derivative and integral of real functions are generally introduced in the last or two last years of secondary school into Italian high schools in a fairly formal way, enriched by technical details and the demonstration of theorems. Results of research on the question "what kind of intuitive ideas are present in the students' minds and how can teaching support or obstruct their development?" were given. The speaker pointed out the presence of propitious intuitions about approximation which often are neglected in didactical practice. A study on epistemological problems in the concept of limit was presented. The interviewee sample encompassed a total of 600 people including students (ranging from 14 to 19 years of age from different types of schools) and adults without specialised mathematical knowledge. The interviewees were asked to describe their ideas on the terms 'limit' and 'infinite'. The natural language register (especially in Italian) does not give a hint to the mathematical meaning of 'limit'. The word limit denotes something which is associated with concepts like 'barrier', 'rule', 'restriction' and other words with an idea of 'finiteness'. Infinity is something that has no limits. On the basis of these findings it appears that the central point is to identify teaching strategies and constructive activities capable of enriching the learning experience and stimulating an evolution of intuitive understanding. An important point is to use approximation as a teaching resource. Students should learn the proper place of empirical methods leading to legitimate approximation schemes which can favour the gradual early development of the concept of limit. It is important to use rich and unusual contexts. The need of approximation can be introduced for example through the question of measuring an area with a curved boundary (a lake, say).

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Education in mathematics – Mathematics in education

The impact of mathematics has been absolutely fundamental to society in the past as well as for the shaping of modern society. Nowadays, not only many sciences, but also society in general rely to an increasing extent on applications of mathematical models. Even though mathematics plays a decisive role in many of the technological advancements in present day society, it is invisible to the general public and difficult to com-

municate. This may provide a platform for tackling the negligence of mathematics by the general public and may lead to actions in order to prevent reductions of the mathematics curriculum in schools. Mathematics education as a scientific subject has emerged in a meeting between two cultures: Theoretically interested educators at the universities and practice-oriented teachers in the school system. There is a great need to relate to the pedagogical and the didactical methods applied in the teaching of mathematics in the many diverse types of educations supported by mathematics. It is important that researchers in mathematics as well as researchers with mathematics education as their speciality participate in developing suitable teaching materials. The involvement of mathematicians is important to ensure the quality of the mathematics taught and specialists in mathematics education are needed to monitor the learning process. Large-scale mathematical experiments can now be performed on the computer. An important task for mathematicians and mathematics educators will be to maintain that mathematics is more than experiments; for a true mathematical education, you need proofs of mathematical results. Quality education of mathematics teachers in primary and secondary schools is the fundamental key to changes and improvements not only in the teaching of mathematics in schools but also, in a longer perspective, for increasing the general level of mathematical knowledge in the population of a country. New media, such as CD-technology and the Internet, will provide many possibilities for valuable mathematical activities, but can never be a substitute for engaged mathematics teachers. And it should never be forgotten that the place where one can really make a difference in shaping the public's attitude to mathematics is by delivering first class teaching of mathematics in schools. Teacher training in mathematics should be given a new impetus!

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Mathematics and mathematics education: A vision of its evolution
Mathematics education is a discipline of knowledge the origin of which dates back to the second half of the 20th century. In general terms it could be described as the study of educational problems linked with mathematical knowledge. During the last decades we have seen university academic activities appear at the heart of the community as mathematics teachers, learners of mathematics and educational mathematicians (corresponding with the term 'Matemática Educativa'). The following questions are the starting point of Cantoral's view: How do new developments in mathematics influence the teaching? How are teachers trained in mathematics? How can mathematicians and educators collaborate? It would be important to see plans and models for this interaction. Three domains must interact: mathematics education as a scientific discipline, mathematics as a scientific domain, and mathematical teaching as a field of practice. To achieve this goal specially designed courses for mathematics teachers are recommended. There are some examples in history for such a fruitful exchange, when Felix Klein and others at the beginning of 20th century had great influence on the changes in curricula. On the other hand the problems in understanding analysis may be related with the development of new models for the 'infinitesimals'.

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Popularization: The case of ill-posed inverse problems
The starting point of this paper is E.C. Wittmann's view of mathematics education as a design science, and in particular what he calls the core tasks of the field. They include:

- Exploration of possible contents that focus on making them accessible to learners;
- Critical examination and justification of contents in view of the general goals of mathematics teaching;
- Development and evaluation of substantial teaching units, classes of teaching units and curricula.

One out of many questions that follow from Wittmann's list is how to find examples of mathematical results which are both 'beautiful' and 'important' and yet can be popularized. Pythagoras' theorem and Euclid's proof of the infinitude of primes are classical examples. There are a few lucky cases from more recent research like public key cryptography, in particular the RSA method. In this example the prerequisites are minimal but the result of a combination of a few elementary though nontrivial and highly powerful mathematical ideas is intellectually amazing, and, as it turns out, of overwhelming practical use. At times the use of metaphors may open a gateway. Here there exists a broad scale of possibilities ranging from metaphors that are quite close to the objects they mimic to more and more remote ones. If used to mimic the system of partial differential equations used for weather predictions, Birkhoff billiards are but a metaphor from a technical point of view. Yet they are reasonably suitable to explain some phenomena of an important nature from a general educational point of view. The extent to which metaphors can help transfer technical mathematical language into semantically available information certainly needs further study. A third approach relies on a process that can be called elementarization. It tries to (re)discover and expose key features of a more advanced topic in a setting that is more easily accessible, i.e. with fewer prerequisites.

As an illustration of the last mentioned approach Kirchgraber proposed to look at so-called 'ill-posed inverse problems.' An inverse problem amounts to reconstructing a cause from its effects. An example is provided by computerized tomography. The ideas behind the solution of ill-posed problems, in particular the concept of 'regularization' (due to A.N. Tikhonov) usually described in a functional analytic setting, can well be illustrated with tools from elementary linear algebra. Due to measurement errors inverse problems suffer from imperfect data. The goal of 'regularization' is to reduce the precision requirements on the data. As a concrete and rather spectacular example the reconstruction of a simple mass distribution from measurements of its gravitational field is presented.

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Similar problems in different contexts: An example from model theory to elementary algebra

Looking for analogies and differences in different situations is educationally and interesting from a scientific point of view. Abstraction itself is based upon the interpretation of analogies and upon the consideration of similar problems in different contexts. The different propensities for self-correction should be considered when we compare researchers and young students. Topics must be considered with respect to their roots and to present-day context. A mathematics educator should be a mathematician, an educator but to some extent also an historian. He or she should be aware of analogies as well

as of differences. This can be illustrated with an example from model theory. Robinson Arithmetic is weaker than Peano Arithmetic. This can be shown by giving an explicit model, namely by taking $Z^*[x]$, the set whose elements are 0 and all polynomials with integral coefficients whose leading coefficients are positive. This model is not isomorphic to the set N of natural numbers (as the standard model for Peano Arithmetic). Clearly, the order in $Z^*[x]$ must be defined in a suitable way. The order is defined in $Z^*[x]$ as follows:

> $f(x) \leq g(x)$ if and only if $g(x) - f(x)$ belongs to $Z^*[x]$ and clearly $f(x)<g(x)$ if $g(x) - f(x) \neq 0$.

It is interesting to confront both models with famous problems in number theory: Fermat's Last Theorem, Catalan's Conjecture (solved recently by to Mihailescu and therefore not anymore a conjecture), and Goldbach's Problem. Interestingly Goldbach's Problem has been solved for non-constant polynomials.

These considerations are connected to fields whose roles in traditional mathematical curricula, referring to primary and secondary schools are weak: mathematical logic and number theory. One must add that even the concept of proof is not known to most pupils.

The presentations were followed by lively, and at some times rather controversial discussions. One central point was the question on whether proposals from research mathematicians could be suitable for teaching in school. However, it should be emphasized that the collaboration of educators and mathematicians is the central message of Theme C 'Mathematics and mathematics education', message that was considered unrealistic by some of the participants in the audience, a view which was, however, refused energetically by some other participants.

This report was written by Fritz Schweiger. He will be happy to be contacted at fritz.schweiger@sbg.ac.at for further information on the work of this Thematic Afternoon.