TSG 8: Research and development in the teaching and learning of number and arithmetic

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Aims, focus, and structure

This Topic Study Group brought together research developments from different countries relating to teaching number and arithmetic, and debated implications for classroom practices. The following three main topics were selected to provide a focus for the presentations and discussions: 1. Developing number sense; 2. Learning arithmetic through problem solving; 3. The role of contexts and models in teaching and learning about number and arithmetic. These topics were first addressed in the plenary talks that started each session. In addition there were refereed papers posted on the ICME-10 website for preliminary reading with authors contributing to discussions. These remain available at www.icme10.dk.

Plenary presentations

In the first plenary presentation entitled 'Developing number sense' *Alistair McIntosh* (University of Tasmania, Australia) elaborated on the ideas expressed in his seminal paper on number sense (McIntosh, A. (1992) A Proposed Framework for Examining Number Sense. *For the Learning of Mathematics, 12*(3):2-8), wherein he provides a conceptual framework for designing and assessing activities around number sense consisting of three components: knowledge of and facility with numbers, knowledge of and facility with operations, and applying knowledge and facility with numbers and operations to computational settings). Basically, McIntosh's paper involved a richly documented analysis of what goes wrong when teaching for number sense is largely absent in an elementary school mathematics curriculum, as well as giving several inspiring examples of instructional tasks and activities that favor the development of number sense. Although progress has been made worldwide, according to this plenary speaker, the basic ideas of the teaching for number sense are not yet (properly) implemented in most mathematics classes.

The next plenary lecture, 'Learning arithmetic through problem solving' by *Christoph Selter* (University of Heidelberg, Germany), distinguished two different types of goals in mathematics education curricula: process related goals (like conjecturing, describing, communicating) and content-related goals (like knowing the facts of the addition table by heart or adding three-digit numbers by means of the traditional written algorithm). According to this author, it is important that neither of the two dominates the other. In his talk he showed how both goals can be integrated in a systematic way. First, he showed why developing a mathematical disposition, including an ability and willingness to engage with mathematical thinking, should be considered as an important goal already in primary mathematics. Then he made some comments on how to learn basic skills (like the multiplication facts) in a meaningful and problem-based context. In the last section of his talk he sketched the approach of the Dortmund MATHE 2000 project, which succeeded in developing teaching/learning units wherein basic skills are not only practised, but also connected with the development of higher order thinking skills. In these units, the learning of the basic skills is treated as an opportunity for children to describe, conjecture and reason: in short, for developing a truly mathematical disposition already from a very young age.

In the third plenary lecture, 'The role of contexts and models' *Brian Greer* (San Diego State University, U.S.A.) took modeling to mean a correspondence established between some aspect of the environment and statements of arithmetic such that relationships within each domain have 'translations' to the other. He distinguished three forms of modeling. A first form where manipulatives and visual representations are models for arithmetic operations. Like several (socio-)constructivist authors e.g (Cobb, Gravemeijer), Greer drew attention to major problems with the use of such manipulations and pleaded for a more sophisticated view of the role of representations. The second form of modeling occurs when existing arithmetical knowledge is intended to be evaluated as an applicable model – or not – for a situation. Here, he pleaded that the essentials of this form of mathematical modeling can already be established early in children's development. Third, Greer introduced the term "developmental arithmetic modelling", to indicate an activity of mathematizing a situation in the course of which enhanced understanding of some arithmetic operation or structure emerges (as in the work of Freudenthal, Gravemeijer, Lesh, and others). This form of activity, stands in contrast with the two previous forms of modeling in which, in the former case, a pre-structured set of physical objects or a visual representation is presented, and, in the latter case, existing arithmetical knowledge is intended to be evaluated as an applicable model – or not – for a situation. According to Greer, this principle of reinventing mathematics through mathematizing a context 'rich and to be structured', as opposed to 'poor and structured', is deeply rooted in the theoretical constructions of Freudenthal and his followers.

Finally, *Julia Anghileri* talked on 'International perspectives on teaching and learning number and arithmetic and future directions'. She outlined some of the expectations in different national curriculum documentation and highlighted potential conflict in teaching for understanding while emphasis actually lies in teaching for fluency in computation. Although there has been a shift to encouraging informal approaches to calculating, she noted that students can undervalue these personal methods when so much time is given to teaching the compact traditional algorithms. She elaborated on research findings that show students need support in developing pencil and paper methods and discussed newer forms of algorithms that preserve number sense while providing a structure for written recording. The major question she identified concerns what we are trying to achieve in teaching calculating techniques for students in today's (and tomorrow's) technological society.

This TSG included also a number of papers refereed by the Organizing Team and circulated through the conference website. These provided a strong background for the discussions at the conference.

Papers discussed during Session 1:
Number sense

Informal strategies and adaptive expertise in proportional reasoning.
Silvia Alatorre and *Olimpia Figueras*, Mexico

Many different number concepts – or one integrated?
Ulrich Christiansen, Denmark

Building and stacking in a count and add laboratory.
Allan Tarp, Denmark

Exploring links across representations of numbers with young children.
Tony Harries and *Jennifer Suggate*, UK

Papers discussed during Session 2:
Learning arithmetic through problem solving

Reinvention revisited learning and teaching decimals as example.
Ronald Keijzer, Frans van Galen and *Lia Oosterwaal*, Netherlands

Two sides of a coin in teaching: An analysis of a lesson on comparing fractions.
Jinfa Cai, USA, *Ida Ah Chee Mok, Agnes Tak Fong Fung*, Hong Kong

The initial use of fractions on adults: the case of Enriqueta.
Marta Valdemoros Alvarez, Mexico

Why 25+4 might be 54: Children's interpretations of uncompleted equations.
Anna Susanne Steinweg, Germany

Papers discussed during Session 3:
The role of contexts and models in teaching and learning about number and arithmetic

Children's strategies for doing simple addition in an instructional environment that favors strategy flexibility.
Joke Torbeyns, Lieven Verschaffel, and *Pol Ghesquière*, Belgium

How basic arithmetic skills are obtained by children with learning difficulties?
Tadato Kotagiri, Japan

Multi-coloured natural arithmetic.
Jean-Noel Manouba, France

An experimental research on error patterns in written subtraction.
Carla Fiori and *Luciana Zuccheri*, Italy

Papers discussed during Session 4:
International perspectives and future directions

Teacher practice and student learning: An 'effective' mental computation lesson.
Ann Heirdsfield, Australia

Teaching Mental Calculation – how successfully are strategies being learnt?
Tom Macintyre and *Ruth Forrester*, UK

Narrowing the gap between mental computation strategies and standard written algorithms.
Ian Thompson, UK

In the final session, during a plenary discussion participants tried to identify the way forward for research over the next four years and looked at possible collaborations that may emerge from the conference. Some of the questions and issues that were addressed during this final discussion are the following:

How important is procedural flexibility or adaptiveness? What is the immediate and long-term effectiveness of aiming for this flexibility? Is it also feasible and valuable for young and weak children? Are there costs of aiming for procedural flexibility? What is the most appropriate way of teaching for procedural flexibility at the elementary school?

Is the problem-solving approach feasible and effective for all children? Do we all agree that a problem-solving approach, which focuses on encouraging students to (re)discover concepts and to (re)invent procedures, is the most appropriate approach to teaching arithmetic in the elementary school? What are the risks of this approach? Is the empirical evidence favoring this approach sound and convincing (even for the weaker students)? Is an evaluation of the value of this approach a (purely) empirical issue?

Is the 'emergent modeling' approach feasible and effective at the elementary school level? As argued convincingly by Greer, the term mathematical modeling is not only used as a synonym of 'applied problem solving'. Besides this type of modeling, which requires that the student has already at his disposal at least some mathematical models and tools to mathematize the problem situation, there is another kind of modeling, wherein model-eliciting activities are used as a vehicle for *the development* (rather than the application) of mathematical concepts. This second type of modeling is nowadays called 'emergent modeling' (Gravemeijer) or, as Greer termed it 'developmental arithmetical modeling'. Although deveopmental arithmetic modeling is getting more and more attention and approval among math educators, it also raises questions about its effectiveness and value. For instance: Do we *always* have to start from informal mathematical activity in real-world contexts in the elementary school?

Arithmetic: a subject for learning mathematics? Elementary mathematics can be viewed and taught as "basic" mathematics – a collection of procedures – or as "fundamental" mathematics. By fundamental mathematics, we mean that it contains already the basis of more advanced concepts and that it forms the foundations for children's further learning of these mathematical concepts. How can we ensure that current innovative approaches that cultivate informal and contextualized thinking, succeed in getting recognition as what we consider as the heart of mathematics (namely a search for patterns and relations) and the instructional attention that they deserve?

This report has been written by Julia Anghileri and Lieven Verschaffel. They are happy to be contacted at jea28@cam.ac.uk and lieven.verschaffel@ped.kuleuven.be for further information regarding the work of this TSG