TSG 10: Research and development in the teaching and learning of geometry

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Aims and focus

The aim of this TSG was to examine and discuss recent research and developments in the teaching and learning of geometry at all levels of schooling from kindergarten to the university, and to present an overview of the current state-of-the-art in geometry teaching and learning. The four sessions held during the Congress, were the culmination of a year long process that we consider an integral part, probably as interesting and rewarding as the sessions themselves, and at the same time the most demanding and delicate. The exchange among the Organizing Team (OT) members and between the OT and the participants was very rich and challenging.

The OT members started by setting their premises: We wanted the TSG10 to be an open and refereed forum for exchange. A call for abstracts was launched by writing and widely distributing a reference paper, proposing themes of interest, and calling for contributions. Fifty-one colleagues responded. Abstracts were reviewed, then a call for full papers launched. A refereeing process was established, by which every paper was reviewed by two OT members. If the paper was to be rejected, a third opinion was sought. At the end, 29 papers were accepted and posted on the website.

The papers were classified under four themes, each theme was studied in one session during the ICME-10 Congress, under the leadership of one or two OT members:

1. Geometry outside the formal Euclidean mould (Leader: *David Henderson*)
2. New views on Dynamic Geometry Software use in geometry classes (Leader: *Iman Osta*)
3. From intuition to formal conceptions in the early grades (Leader: *Ewa Swoboda*)
4. Developing mathematical thinking and attitudes through secondary or college level (Leaders: *Verónica Hoyos* and *Harry Silfverberg*).

The theme leaders wrote, as well, papers setting the stage for the theme discussions and summarizing the papers corresponding to their theme. In order to allow for all papers to be presented, and for rich discussions, the OT opted for short presentations (5-6 minutes each) in 4 panels, followed each by 20-25 minutes of discussion. A note was sent to all participants, encouraging them to read all the papers (made available on the website) before the Congress. In addition to the theme leaders' papers, only 23 out of 29 accepted papers were actually presented. The other six papers' authors did not attend the Congress.

Following is a summary of the proceedings and the main ideas discussed in the four sessions:

Session 1: Geometry outside the formal Euclidean mould

In his paper "Geometry Outside the Formal Euclidean Mould", *David Henderson* (USA) looks into the origins of geometry, and identifies four historical strands, from which it emerged: Art/Patterns, Navigation/Stargazing, Building Structures, and Machines / Motion. The paper further traces the evolution of geometry and mainly the foundation of non-Euclidean axiomatic systems. It also traces Hilbert's and others' developments of more complete and abstract sets of axioms for Euclidean geometry, yet not ignoring "the tendency toward intuitive understanding". Challenging questions arose, such as "Do we still need to confine our geometry curricula to Euclidean geometry?"

Taro Fujita, *Keith Jones* and *Shinya Yamamoto* (UK and Japan) in their paper, "The role of geometrical intuition in the learning and teaching of geometry", argue that geometrical intuition is important – as a skill to create and manipulate geometrical figures in the mind, to see geometrical properties, to relate images to concepts and theorems, and to decide where to start when solving geometrical problems. Intuition is also identified by *Gloriana González* and *Patricio G. Herbst* (USA) in their paper, "Competing discourses defining the geometry Course: What was new in the twentieth century?", as they report that in the discourses in North America that defined school geometry courses there were four main arguments, including: "geometry as an opportunity to experience the work of doing mathematics, including the use of proof", a formal argument: "geometry as a case of logical reasoning", and a utilitarian argument: "geometry provides tools for future work and applications".

In the discussions, attempts to use simplified axiom systems in order to make the ideas more easily accessible than is possible using the full set of Hilbert's axioms for Euclidean geometry were presented. *Milan Hejný* and *Naďa Stehlíková* (Czech Republic) in their paper "Didactic simulation: Approaching deep ideas in geometry", investigate the use of "trileg mini-geometry", which has a simplified axiomatic structure, as a way to make these foundational ideas accessible to prospective teachers.

Among the papers connecting geometry to other mathematical disciplines, was "A geometry lesson using signed area" by *C.T. Zahn* (USA) who develops an extended geometry lesson that introduces middle school students to the connections between Euclidean geometry, computational geometry, analytic geometry, and algebra. *Nina Hayfa* (Lebanon), in her paper about the learning of vectors (one of the major concepts that link geometry to algebra), "Impact of the language on the conception of the vector", points out that the language used to describe vectors in textbooks and in the classrooms, causes confusion between bound vectors and free vectors.

Session 2: New views of Dynamic Geometry Software in geometry classes

Dynamic Geometry Software, DGS, had an important share in the discussions. Under the title "New views on DGS use in geometry teaching / learning", *Iman Osta* (Lebanon) opened the discussion by asking some questions that mark a shift in the type of issues related to DGS: How does DGS use in the classroom influence the debates between intuitive and formal geometry advocates? How are the DGS visualization capabilities affecting the necessity (or the opposite) of providing formal proofs? Does the use of DGS in the classroom create new types of geometric reasoning? How can DGS environments be compared to other mediating tools in the teaching of geometry? How would geometry curricula be modified to integrate the use of DGS?

Some of the papers which involved DGS issues also addressed the theme of geometry outside the Euclidean mould. Even though DGS was originally developed for the investigation of Euclidean geometry, they can also be used to study non-Euclidean, spherical, and hyperbolic geometries. *Bjørn Felsager* (Denmark) in "Introducing Minkowski-geometry using dynamic geometry programs" demonstrates that DGS can be used to give a non-axiomatic approach to teaching the non-Euclidean Minkowski geometry. Along the same line, *Margaret Sinclair* (Canada), in her paper "Adopting Cinderella and Spherical Sketchpad as exploratory tools: Some reflections on motivating factors", examines students' comments and assignments, in a graduate geometry course, who chose to utilize Cinderella or Spherical Sketchpad as additional exploratory tools outside of class.

DGS create computer microworlds with Euclidean geometry as the "embedded infrastructure". Nevertheless, *Francis Lopez-Real* and *Allen Leung* (Hong Kong) in their paper, "The conceptual tools of Euclidean and dynamic geometry environments" demonstrate that the function of "dragging" is a powerful tool in DGS that does not have a formal counterpart in Euclidean geometry. "Dragging" seems to be a conceptual tool that is, to the learner, as legitimate as the traditional Euclidean tools of compass and un-marked straightedge. The authors ask: Can we expand the usual formal Euclidean axiomatic system to include dragging?

Jeff Connor, *Laura Moss* and *Barbara Grover* (USA), in their paper "An obstruction to exploration with Dynamical Geometry Software" try to investigate whether or not students made effective use of DGS to explore the validity of geometrical statements, using Sketchpad. The analysis indicates that the way students regard the definition, whether as a 'dictionary definition' or a mathematical definition, affects the use of DGS. The effective use of DGS is also influenced by the ability to correctly parse a mathematical statement. *Thomas Gawlick* (Germany), in his paper "Restructuring dynamic constructions: Activities to stimulate the development of higher level geometric thinking", presents a sequence of tasks designed for student teachers using DGS. The aim is to reach higher level thinking. He relates DGS properties with the transitions through a revised version of the van Hiele levels, based mainly on an interpretation by Freudenthal.

Regarding teachers' use and attitudes toward DGS, *Lil Engström* (Sweden), in her paper "Examples from teachers' strategies using a dynamic geometry program in upper secondary school", presents examples of teachers' strategies when using Cabri. The assumption is that teaching strategies might depend on the teachers' definition of mathematics, on how they perceive the concepts of learning and knowledge, and on their experience of the computer program and experience of teaching.

Session 3: From intuition to formal conceptions in the early grades

Two of the main themes discussed in TSG 10 dealt with the learning of geometry at different levels, using various approaches. Reflecting on the teaching of geometry at an early age, *Ewa Swoboda* (Poland) stresses, in her paper "From intuitions to formal conceptions", the role of intuition in the teaching/learning of geometry, whether in the creation of the geometrical world that emerges from the real world, or in understanding space and relations between figures as a dynamic space organization. She identifies geometry as a way of persuing long-term aims related to the philosophy of mathematical thinking: from perception to definition and mathematical formulation; and for finding the general in the particular. *Mariolina Bartolini Bussi*, *Maria Alessandra Mariotti* and *Franca Ferri* (Italy) stress, in their paper "Semiotic mediation in the primary school", the opinion that the presence of artefacts does not mechanically determine the way in which they are actually used and conceived of by the students. They distinguish between (at least) two types of artefacts: a primary artefact (e.g. concrete instrument handled in the solution of problems) and a secondary artefact (e.g. text or system of signs). Their research hypothesis is that the intrinsic polyphony of the artefacts supports the production in classroom activities of the polyphony of voices (forms of speaking and thinking).

Based on Fishbein's "theory of figural concepts", *Edyta Jagoda* (Poland) conducts experimental work to investigate children's intuition of mirror symmetry in a plane. Her main research aim, as expressed in her paper "Perceiving symmetry as a specific placement of figures in the plane by children aged 10-12", is: How does the children's perception of the relationship between one shape and its transform shape in the plane, contrast with their noticing the dynamism of the transformation. Under the same topic of transformations, *Charlotte Bouckaert* (Belgium) describes a proposal (among others) to approach the notion of orientation of plane or space. In her paper "Some aspects of transformation geometry in primary school according to Michel Demal", she presents Demal's way of using a spiral curriculum in the spirit of Jerome Bruner. By comparing two figures and by using transparent sheets, children begin to become familiar with transformations.

Paola Vighi (Italy), in her paper "The geometry of squared paper" describes results of using squared paper for a task of drawing isosceles triangles. She found that, though the grid might help children, it might as well interfere with what the pupils have in mind with regard to isosceles triangles and act as an obstacle, thus making the task more difficult.

The emergence of the geometrical world from the physical world and from various activities was frequently visited. *Nancy Vezina* and *Lucie DeBlois* (Canada), in their paper "Geometry in context at the primary level: Using the environment as a starting point", suggest that the living environment can be used as a starting point for teaching about different geometrical shapes. They found that through different learning activities using the environment, children create and use a wide range of procedures that differ from those that are usually developed at school. In the same spirit, in the paper "Drama in teaching and learning geometry", *Asuman Duatepe* and *Behiye Ubuz* (Turkey) suggest that drama creates an environment in which students construct their own knowledge by means of their experiences. By using this method students build their meaning of a word, a concept, or an idea. In the paper "Geometrical pre-conceptions of 8 years old (third grade) pupils", *Carlo Marchini* and *Maria Gabriella Rinaldi* (Italy) use two visual representations for the concept "isosceles triangles": as a flag, and as a roof; saying more precisely that they use drawing "orientation" in the perception of "isoscelity" of triangles. They test how those orientations have an impact on children's mental representation for isosceles triangles.

We can see "reality" in a larger perspective. It could mean a comfortable environment, which can support mathematical thinking processes. For some students, a rigorous way for thinking, necessary for making mathematical proofs, is too difficult. They need to have some support by using very familiar facts, which they can imagine, draw, etc. *Michael Koren* and *Dan Amir* (Israel) in their paper "The rectangular approach – A royal road to Euclidean geometry in intermediate school" create such an environment for teaching Pythagoras' theorem and its proof.

Session 4: Developing mathematical thinking and attitudes through secondary or college level

Verónica Hoyos and *Harry Silfverberg* (Mexico and Finland) state, in their paper "Developing mathematical thinking and attitudes through secondary or college level", that many mathematicians consider geometry to be one of the branches of math least contaminated by rules, formulas, or algorithms in discovering and solving problems. Yet they recognize this feature as being problematic for the teaching and learning of geometry, especially at the level of deductive reasoning, a main area of current research. This paper identifies some of the major theoretical perspectives which serve as a background to current research in didactics of geometry (Piaget, van Hiele, cognitive science, and more recently socio-cultural approaches, using the theoretical constructs of Vygotsky and his followers). This line of research continues with the incorporation of new technologies, like dynamic software of geometry and internet chatting.

In his paper "Describing undergraduates' geometric thinking via an "object of thought" interpretation of the van Hiele model", *Stephen Blair* (USA) presents a review of the van Hiele model and its evolution through research works. He uses the "object of thought" interpretation of the model to describe undergraduates' geometric thinking and use of definitions across three different geometries (taxi-cab, spherical, and Euclidean), while documenting transitions across levels 3, 4, and 5 of the model.

Jaguthsing Dindyal's (Singapore) paper, "Students' thinking in school geometry: The need for an inclusive framework", raises issues about geometric thinking and the need to conceptualize it within a broad framework. The paper is the outcome of a piece of research investigating students' use of algebraic thinking in geometry, while solving problems which involve, among other things, the use of variables and unknowns, writing and solution of simple linear equations, and recall and use of formulae within geometry. Three aspects of algebraic thinking were investigated: the use of symbols and algebraic relations, the use of representations, and the use of generalizations within geometrical contexts.

Through placing students in a problem-situation, *Naim Rouadi* (Lebanon), in his paper "The development of geometrical thinking of Lebanese students aged 11 – 15", focuses on the geometrical thinking of Lebanese learners, taking into consideration van Hiele's five levels. The paper documents how the analytical perception of the problem allows the learner to move from drawing to figure (as a mathematical model) and between two registers: from geometry to arithmetic.

Oleksiy Yevdokimov's (Ukraine) paper "Skills of generalization in learning geometry. Are the students ready to use them?" tries to study the generalizing difficulties faced by students. It identifies three types of generalizations: generalization of definitions for different geometrical objects, generalization of geometrical object's properties by giving up certain features, and creative generalization.

Conclusion

The above survey of research studies, reflections and discussions raised in the TSG 10 on geometry shows the richness of this field, the various orientations and perspectives under which the teaching/learning of geometry can be studied. Richness, yes, but maybe fragmentation and lack of focus, as well. Beginning 2005, the NCTM's Research Committee presented a call for an Agenda for Research Action in Mathematics Education.

Now may be the time for us to define for ourselves some key issues for joint research and to consolidate the community that investigates the teaching and learning of geometry.

This report was written by Iman Osta and reviewed by the OT members of TSG 10. The author is happy to be contacted at iman.osta@lau.edu.lb for further information on the work of this TSG.