TSG 16: Visualisation in the teaching and learning of mathematics

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Aims and focus

It has come to be recognized that visualization and visual imagery are important aspects of mathematical understanding, insight and reasoning, and that visual presentations as well as attention to students' diverse use of visuals are essential to effective mathematics teaching. This topic study group held a discussion of visualization from multiple perspectives, addressing a variety of questions.

What are the roles of visuals and visualization in mathematics? How do visual forms and visual reasoning about mathematical ideas affect diverse mathematical fields, historically and at the present time? What are some of the classic and most effective examples that can illustrate these roles for students?

What is the psychological role of visual thinking, and related forms of representation (e.g., spatial and kinaesthetic representation), in the learning of mathematics? How do experts and novices "learn to see"? Do mathematicians, teachers and diverse students "see" different things when working with the "same" diagram or sketch? How can this be ascertained?

What do studies of cognition and diagrammatic reasoning tell us about visual representation in the human brain? How can we teach and learn to use visualization more effectively?

What relation now exists or should exist between visual forms and visual reasoning, and the mathematical curriculum? What kinds of external visual representations and internal visualizations (mental imagery) occur as children build concepts in relation to particular mathematical topics and processes, such as whole numbers, proportional reasoning or fractions? How can appropriate visualization increase mathematical power?

How does visualization relate to other ingredients of mathematical understanding, such as the use of symbolic notation? How do various visualizations relate to students' affect and motivation in relation to mathematics? What distinguishes effective from ineffective use of visuals in the classroom?

What are some of the most effective technology-based tools for mathematical visualizations: for example stills, sequences of stills, animated visuals, 2-D and simulated 3-D, passive and interactive? How are these used in the practice of mathematics? How can these be used most effectively in mathematics teaching?

The program included a balance of oral presentations and discussions, structured discussions of sets of distributed papers, and open discussion among participants. The papers whose descriptions follow were presented in person, for 15 minutes, with 5 minutes for discussion. Related papers were circulated for discussion, through the internet and at the sessions themselves.

In addition, the two co-chairs presented "book-end" initial and final talks, to introduce some conceptual terms and frameworks (Goldin) and to provide some summary reflections and suggestions (Whiteley).

Session 1

The first day, after initial introductions, *Gerald Goldin* opened with an overview of some aspects of visual imagery and cognitive representation, in relation to mathematical thinking. Key ideas include the notion of "pattern", internal vs. external systems of representation, and the different types of internal representation that are possible: verbal/syntactic, imagistic (including visual), formal notational (symbolic), heuristic and executive, and affective (emotional). Internal visual imagery can stand for (or represent) external objects or diagrams, as well as other kinds of internal configurations – e.g. words or formulas. Furthermore, representing can be a two-way relationship. Children display a rich variety of imagery, evident in their descriptions and drawings (Thomas, Mulligan, & Goldin, 2002; DeWindt-King and Goldin, 2003).

There followed a discussion of the related paper, "An investigation of the cognitive processes required for a Mathlet", *Ozlem Ceziturk*, Bogazici University, Istanbul, Turkey.

Session 2

The second day began with an animated display by Michela Maschiatto, anticipating questions to be raised in her talk. The session then featured two presentations, followed by open group discussion:

- "Can visualization promote causal thinking?", *George Malaty*, University of Joensuu, Finland. This paper describes some alternative forms for visualizing arithmetic processes involving fractions. A decomposition is presented of the usual single diagram into several steps, with corresponding arithmetic steps; and the ability of students to generalize the pattern is described.
- "Developing pictorial ideas in learning numbers and calculations", *Tadato Katagiri*, University of the Ryukyus, Japan. This paper describes a long-term study of the visual forms used by brain injured and other developmentally delayed students. One-on-one tutoring and encouragement to draw and play are used to develop arithmetic sense with these students. Examples of the evolutions of the diagrams are presented; more material is available on the TSG website, www.icme10.dk.

Session 3

On the third day, two additional presentations were followed by discussion of two additional related, posted papers:

- "Visual representation in the construction of mathematical meanings", *Michela Maschietto, Maria G. Bartolini Bussi, Maria Alessandra Mariotti,* and *Franca Ferri*. This paper, presented by Michela Maschiatto, University of Modena, Italy, presents a project in which students work on the mathematics of perspective drawings, using classical writings, reconstructions of classical apparatus for drawing ("intersecting the visual pyramid with the picture plane") and photo realistic computer animations illustrating such instruments. The project investigates students' ability to understand the process of drawing in perspective, and to describe the functioning of other instruments designed for perspective drawing presented in photo realistic animations.

- "Mental model training wheels: Scaffolding mental imagery with partial sensory support", *Glenn Gordon Smith* (USA) and *Jim Morey* (Canada). A series of experiments on spatial visualization, using specially designed software, were presented. Among other things, these studies showed that students who were actively engaged in an activity learned more about the specific task, than students who observed. Students who had alternating roles over the same time, learned the same amount as those active for the whole period. The most recent study investigated imagining the 'stamping' face of an illustrated cube which was rotated in pre-assigned steps then placed down onto the page. The variable was whether operations were better internalized when some of the visual cues were hidden – but no significant difference has yet been detected.

There followed discussion of two related papers: "Towards a theory of visualization by dynamic geometry software: Paradigms, phenomena, principles", *Thomas Gawlick*, (Germany) and "Students' development of geometrical concepts through a dynamic learning environment", *Isil Ustun* and *Behiye Ubuz*, Middle East Technical University, Ankara, Turkey.

Session 4

On the final day, three additional presentations were followed by a general discussion for the 30 minutes.

- "Dynamic geometry software as a simulation tool for algebra problems", *Stefan Halverscheid*, University of Bremen, Germany. This paper presents a project in which students can choose between using a CAS software (*Maple*) or a Dynamic Geometry Software to investigate and illustrate some problems of transforming and solving linear equations in two variables. In both situations, the students use pointing at, or referring the diagrams to express ideas and reasoning. With the dynamic diagrams, more active words were added to the descriptions. Some of the strongest students, realizing both tools can be used, select the dynamic geometry environment, for the interaction of the diagram with the algebra, where the CAS system offered only entry into the algebra to generate the diagrams.
- "The problem of misperception in mathematical visualisation", *John Malone, Daniel Boase-Jelinek, Martin Lamb, Sam Leong* (Australia). Presented by John Malone, Curtin University of Technology, Australia, and Martin Lamb, this paper describes a study of students' work on a task involving rotation and reflection of a plane figure, and identifiable patterns of misperception of the impact of such a transformation, something which is distinguished from misconceptions more traditionally studied in mathematics education. The impact (or lack of impact) of interventions with software and one-on-one work are also investigated, demonstrating the persistence of some of the identified misperceptions.

Some responses and reflections: *Walter Whiteley* extracted some key points and issues for further work from the authors' circulated papers, and from several of the regular presentations (Dörfler, Mason, Nuñez). Specifically highlighted were points such as the fact that different people see and attend to different things in the same diagram or sequence, as well the urgent need for teachers to be aware of possibilities for use of variable visualizations by themselves and their students and to be able to consider what various students are "seeing". The group considered the possible value of forming a community focusing on visualization, with some shared materials, assumptions, questions, etc.

In the concluding discussion, there was agreement that mathematicians at all levels do make effective use of visual representations and reasoning. For some, including those in modern research, the visual work is essential. Thus it is equally essential that the teacher have access to and value multiple visual forms. The current tendency to devalue visualization in comparison to symbolic computation was discussed.

Students have different strengths and weakness in learning styles, and equity requires that students have the option of building on their visual strengths and having these contributions valued as significant mathematics.

Finally an informal invitation was given for participants to join a continuing electronic discussion (organized by the second co-chair). Further invitation will be made to all participants who signed on to the Topic Study Group's list.

Other circulated and referenced papers:

Adrian M. DeWindt-King and *Gerald A. Goldin* (2003), "Children's Visual Imagery: Aspects of Cognitive Representation in Solving Problems with Fractions". *Mediterranean Journal for Research in Mathematics Education* **2** (1), 1-42.

Noel D. Thomas, Joanne Mulligan, and *Gerald A. Goldin* (2002), "Children's Representation and Structural Development of the Counting Sequence 1-100". *Journal of Mathematical Behavior* **21** (2002), 117-133.

Walter Whiteley, "Visualization in Mathematics: Claims and Questions towards a Research Program", 2004, www.math.yorku.ca/~whiteley.

This report was written by Gerald A. Goldin and Walter Whitley. They are happy to be contacted at geraldgoldin@dimacs.rutgers.edu and whiteley@mathstat.yorku.ca, respecttively, for further information on the work of this TSG.