TSG 19: Reasoning, proof and proving in mathematics education

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This TSG included eight presentations given by:
Kirsti Nordström (Sweden), *Lara Alcock* (United States), *Susanna Epp* (United States), *Raisa Buberman* and *Marita Barabash* (Israel), *Virginie Deloustal-Jorrand* (France), *Alexander Khait* (Israel), *Takeshi Miyakawa* (Japan and France), and *Manya Raman* (United States).

Aims and focus

The aim of TSG 19 was to provide opportunity for TSG participants to share their research in reasoning and proving in mathematics education, with particular focus on:

- An international perspective on research on reasoning and proof, with particular emphasis on research conducted outside of Europe and North America
- Transition from informal argumentation to formal proof in mathematics classrooms, including classrooms where technology is used.

A summary of the eight presentations (listed in the order the papers were presented).

Lara Alcock, "Mathematicians' perspectives on the transition to formal proof".
This talk was based on interviews with mathematicians who teach a transition course at a large state university in the US. Based on their comments, Alcock identified four modes of thinking that are used flexibly by successful provers. These were: instantiation, structural thinking, creative thinking and critical thinking. Alcock illustrated each of these modes and identified an emphasis in current teaching on developing skills associated with structural thinking.

Kirsti Nordström, "A pilot study on five mathematicians' pedagogical views on proof".
Nordström has analyzed five mathematicians' pedagogical views on proof in order to test and improve the conceptual frame that she has created on the basis of the literature. The frame with three categories, conviction/explanation, inductive/deductive reasoning and aspects of formality, the level of rigor and the language, was developed to identify features in mathematicians' utterances and to consider the utterances in their social, historical and cultural context. During the process of analysis of the five interviews several new aspects emerged, for example the aspect of transfer Nordström considered proof as an artefact – a resource for mathematical learning. She argued that Lave and Wenger's concept of transparency captures a dual function of proof as a learning resource in mathematics: It needs to be both seen (be visible) and to be used and seen through (be invisible) to provide access to mathematical learning. Students' access to proof is a central issue for her study.

Alexander Khait, "Proofs as a tool to develop intuition".
Traditionally mathematics for non-mathematicians (e.g. physicists and engineers) is presented on an intuitive basis. Usually the formal side of mathematics is downplayed. If proofs are taught to this population at all, the teaching proceeds from non-formal explanations, banking on students' intuitive understanding. As a result of the computer revolution there is a large increase in individuals who are not inclined to study mathematics but need it for professional activities, typically connected to various computer applications. In this context the possibility of being satisfied with intuitive explanations and understandings is unacceptable: a computer can be engaged only in a formal talk. Computer professionals have to be comfortable with formal definitions. While programmers usually do not prove the correctness of their algorithms, to become a good programmer one has to develop intuition to create correct programs. This implies a new role for proofs, namely transitions from formal to informal proofs, teaching students to distinguish intuitively between true and false propositions. Khait discusses a teaching design aimed at achieving this purpose.

Manya Raman, "Key ideas in the context of a proof from collegiate calculus".
Raman's talk centers around a newly proposed theoretical model of mathematical proof. The model accounts not only for what a proof is but also how one gets created. This model grew out of empirical research in which college freshmen, graduate students, and mathematics faculty were asked to compare different proofs of a claim from college calculus. It appeared that both the novices and the experts distinguished between an essentially public aspect of proof (the formal, rigorous aspect) and a private aspect (the informal, intuitive aspect). The difference between those who were mathematically sophisticated and those who were not was that the former saw connections between the public and private domains while the latter did not. The link between the public and private domains is called the "key idea" of the proof. In the talk Raman defined "key idea" and gave examples across a fairly broad range of mathematical proofs. The next step in Raman's research is to explore ways in which the "key idea" could be used as a pedagogical tool in helping students understand given proofs and produce proofs of their own.

Susanna Epp, "The role of logic in teaching proof".
Susanna Epp's presentation argued that even simple mathematical proofs and disproofs are more logically complex than most mathematicians realize, and it discussed two possible reasons why so many students have difficulty with proof and disproof: differences between mathematical language and the language of everyday discourse, and the kinds of shortcuts and simplifications that have been part of students' previous mathematical instruction. It described research about whether explicit instruction can help students develop formal reasoning skills and suggested that such instruction can be successful when there is appropriate parallel development of transfer skills, such as the use of exercises to express statements both formally and informally and overt reference to logical principles in later mathematics instruction.

Raisa Guberman and *Marita Barabash,* “Improving reasoning abilities of 5th-6th grade pupils using a specially designed teaching unit in pre-formal logic”.
Numerous researchers in mathematics education have referred to the need for intermediate stages towards formal proof and reasoning in mathematics. Guberman and Barabash assert that the idea of pre-formal proof must include pre-formal logic. In the course of school teaching and learning the necessary linguistic skills and thinking abilities are not being sufficiently developed. This in turn causes essential difficulties when a student arrives at the deductive stages of mathematical learning, e.g. in deductive geometry. Keeping in mind the purpose of developing the pre-formal logic in primary school pupils, a group of math educators from the Achva College in Israel have developed a teaching unit named “Learning with Alice to Think and to Reason” intended for the pupils of 5th – 6th grades of primary school. Based on this unit, Guberman and Barabash have planned an experiment to assess the effect of teaching in thus designed logic environment, on the development of pupils' ability to reason logically and to build logically valid argumentation. They presented the results of this experiment and some preliminary conclusions.

Virginie Deloustal-Jorrand, “Polyminoes: A way to teach the mathematical concept of implication”.
Three points of view on implication were presented by Deloustal-Jorrand: a formal logic point of view, a deductive reasoning point of view, and a sets point of view. From the formal logic point of view, "if A then B" normally simply means "not A or B". From the deductive reasoning point of view, the statements "A is true" and "if A then B" force the conclusion “B is true”. And from the sets point of view, “if A then B” means that the set defined by A is a subset of the set defined by B. Deloustal-Jorrand's study had as its research hypothesis that it is necessary to know and establish links among these three points of view to have a good understanding of implication and to use it correctly. She gave beginning mathematics teachers problems to solve that involved statements about very concrete situations – “paving” various types of polyminoes by dominoes. The first set of questions required students to make deductions; the second asked them to criticize proposed “proofs” about the polyminoes. Although the analysis of the results of the study is incomplete, the problems presented were clever and worth using as exercises in transition-to-higher-mathematics classes or in discrete mathematics classes.

Takeshi Miyakawa, “The nature of students' rule of inference in proving: The case of reflective symmetry”.
Miyakawa gave some students two related problems about reflective symmetry in geometric figures. The two problems had opposite answers, and Miyakawa discovered that the students generated incorrect “rules” to justify some of the steps in their "solutions". The students themselves generate a rule of inference. For some of the students the backing they have for validating a rule of inference is construction (e.g., “if one can construct a figure which is accepted visually or perceptively, these properties can be a conditional statement of the rule of inference”). The construction is a way to validate the rule. However, one might raise the following problem: it is not certain whether the rule validated by construction will be accepted by the theory admitted at the beginning. Mathematically or theoretically the rule of inference should be accepted by the theory admitted at the beginning. But, are the rules of inference used in mathematics always

as such? It seems that there is not sufficient attention to whether the rule is accepted by the theory or not. This reflection poses an educational question: "What should students' rule(s) of inference rely on?"

Questions for further considerations:

- Alcock's four categories seem related to ways of thinking that have been discussed in other terms in the research literature. How do these categories differ?
- Nordström's work shows that mathematicians and students have very different ideas about what the other knows. What would be the effect if mathematicians knew more about students, or student knew more about mathematicians, before beginning university studies?
- Khait's presentation on proof for computer science students raises the question about "proof reading" versus "proof writing." A possible research question would be whether proof reading is conceptually prior to proof writing, and how one learns to read a proof.
- Raman's distinction between private and public proving is worth wider study. For example, being (personally) convinced and a fact being (publicly) verified are different but related events. Both are central to proving, but sometimes only one is considered, or the two are treated as equivalent.
- Epp's presentation brought out an important distinction between learning algebra and learning proof. Algebra can be seen as generalized arithmetic. But logic is not viewed by many as generalized speech. What implications does this have for teaching logic?
- Guberman and Barabash's presentation raised the issue of what is natural about logic. What is "common sense"? Logic is only generally functional within mathematics, in other contexts meaning matters more than logic. And yet logic is useful in culturally specified ways. How has logic evolved as a way of thinking in an illogical world? How might it develop in the worlds of children? How can the culturally specific use of logic in mathematics be taught?

This report was written by Guershon Harel with the support of Susanne Epp and David Reid.
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