TSG 22: Learning and cognition in mathematics: Students' formation of mathematical conceptions, notions, strategies, and beliefs

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Introduction

TSG 22 focused on four main aspects of learning and cognition in mathematics:

1. Models of mathematical thinking and understanding;
2. Learning and instruction: the role of technology;
3. Teachers' cognition and students' formation of mathematical conceptions;
4. Difficulties in learning mathematics.

Each session was devoted to one of these issues. We shall briefly describe the main themes that were raised in each session. A short summary of each of the ten invited talks are provided, as well.

Session 1. Models of mathematical thinking and understanding

In this session *Anna Sierpinska* (Canada), *Celia Hoyles* and *Richard Noss* (UK) and *Günter Törner* (Germany) presented and discussed three main current issues in mathematics education: The complementary roles of theoretical and practical thinking in mathematics learning, the notion of situated abstraction and its relevance to mathematical understanding and the epistemological dimensions for knowledge systems of teachers as reflected in the case of linear functions. The three presenters highlighted the need to develop new and more robust paradigms for thinking about the development of mathematical thinking and understanding, while emphasizing the following aspects: the nature of mathematics, the role of the teacher, the complex process of instrumental genesis and the connection of tool use and traditional techniques. This session provided the setting for sessions 2-4.

Sierpinska, in the beginning of her talk, "On the necessity of practical understanding of theory" raised a crucial issue: What is the use, for didactics of mathematics, of *general* models of mathematical thinking and understanding, which are not specific to some concrete mathematical contents? She took the example of the classical epistemological distinction between theoretical and practical thinking: a crude model of an extremely complex reality. Sierpinska argued that more refined models are needed, and even these could be uninformative unless complemented by analyses of the particular mathematical content. She presented examples of students' work in linear algebra, showing that successful students certainly have a sense of what it means to work within a theoretical system, but these students are also very "practical" in moving about in the theory, finding conceptual shortcuts, and picking exactly what is needed from the theory. Moreover, these students readily give up on rigor and generality of the solution, if this is not absolutely necessary for obtaining a solution. They solve the problem at hand; they do not develop a theory of solving all problems of a kind. Yet there is an undercurrent of generalizable techniques in their solutions. Sierpinska concluded by emphasizing that using only the theoreticcal/practical distinction does no justice to students' ingenuity and that it is necessary to get into the particular mathematical contents of the problem.

In their talk "Situated abstraction: mathematical understandings at the boundary", *Hoyles* and *Noss* argued that any model of mathematical thinking and understanding needs to take account of two consistent research findings: (1) mathematical knowledge is generally characterized by fragmented and pragmatic strategies focused on the task in hand, and (2) learning mathematics is neither necessarily cumulative nor is it necessarily portable to novel situations. They showed on the basis of research in two widely divergent settings (children's use of computational systems and adults' use of mathematics in workplaces in which the level of expected mathematical knowledge is rather ill-defined) that tools and representational infrastructures shape the nature of mathematical knowledge. Further, they illustrated how conceptions of mathematics were situated in terms of language and connectivity with context and, at the same time, abstract – in that the representations extended beyond the immediate to take account of more general mathematical structures. They discussed the notion of *situated abstraction,* which seeks to describe how the process of generating and expressing meanings with the available representational infrastructures tends to produce individual and collective understandings and ways of working that appear divergent from standard mathematics: the 'symbolic tools' used are *constitutive* of meaning and, by implication, of thought. The presenters also suggested that the notion of situated abstraction could be defined in a broader theoretical context, e.g., within activity theory and building on the notion of boundary object: For example, it is at the boundaries between activities that communication of meaning and 'transfer' can be problematic unless and until the different conceptualizations – and the language in which they are expressed – are brought into alignment.

Törner: "Epistemological dimensions for knowledge systems of teachers – the case of linear functions". Törner argued that even though an immense and very diverse literature exists about the knowledge of mathematics teachers, valuable classification systems, which are both universal and suitable for specific contents, are still lacking. He noted that such a multidimensional classification system should take advantage of Shulman's categories of subject matter knowledge and pedagogical content knowledge and integrate those aspects. He emphasized that, in order to present pragmatic, workable schemes, it is necessary to restrict the focus on the epistemological type of knowledge only. He then reported on investigations on linear functions (Grade 7) and the specific knowledge and belief systems of teachers concerning this area. The discussion was based on data gathered within a recent bilingual German/Dutch in-service-training project integrating video-lessons in the two countries and discussions between the teachers.

At the end of this session seven short papers were distributed among the participants. These papers focused on two main issues. Four papers dealt with students' formation of mathematical concepts, including whole numbers (*Liu Jing* and *Song Nai-qing*, China), fractions (*Suhaidah Tahir* and *Md Nor Bakar*, UK) and functions (*Pal Lauritzen*, Norway, and *Jonathan Stupp*, Israel). Three described and discussed issues of communication in mathematics (reading problems in mathematics (*Hak Ping Tam*, Taiwan) tacit-explicit perspective for the cognition in school mathematics (*Cristina Frade*, Brazil) and deaf children's concept formation in mathematics (*Elsa Foisack*, Sweden)).

Session 2. Learning and instruction: the role of technology

In this session *Ricardo Nemirovsky*, *Tracy Noble*, *Cara DiMattia* and *Apolinario Barros* (USA), *Colette Laborde*, (France), and *Kenneth Ruthven* (UK) presented their views regarding the role of technology in learning and cognition in mathematics. The researchers defined, discussed and contrasted psychological, philosophical, systemic and instructional issues related to the role of technology in mathematics education.

Nemirovsky et al in "Manipulatives, Limit Objects, and Mathematics Learning". addressed two questions: 1) If mathematics education aims at familiarizing students with abstractions, and abstractions cannot be directly touched, seen, heard, etc., why would bodily activity be relevant to learning about mathematical abstractions? And 2) How and why should the use of tools which engage eyes and hands in drawing, writing, manipulating, or touching be relevant to learning about mathematical abstractions? The presenters advocated a view of doing and thinking as woven in body activity in all its forms: eye motion, drawing, writing, grasping, gesturing, talking, and so forth. They showed a classroom episode that took place in a public high school in Boston. The students were using a "Drawing Machine" device with which two students can jointly draw a figure by hand-controlling the X and Y axes. This machine produces also a graph of the motion along the X and Y axes over time. The latter corresponds to two parametric functions for the figure produced. The presenters concluded that through tool-use students develop specialized and bodily sensitivities and goal-oriented responsiveness which allows them to imagine necessary conditions for a mathematically-defined trajectory.

Laborde's talk "New technologies as a means of observing students' conceptions and making them develop: the specific case of dynamic geometry" started from the hypothesis that solving mathematical tasks in a technological environment requires two kinds of knowledge, mathematical and instrumental. Most of the time, especially because ICT used in the teaching of mathematics embeds mathematics, both types of knowledge interact in the use of technology, giving rise to what Rabardel calls instrumentation schemes. This interaction can be used to favour mathematical learning. Sequences of tasks are designed in the computer environment. The interventions of the teacher are critical to establish a correspondence between the actions in the environment and the theoretical concept to be learned. These interventions can support an internalization process (in the Vygotskian sense) transforming actions made in the computer environment into mathematical knowledge. Examples showing the interplay between mathematical and instrumental knowledge in sequences of tasks were shown in the case of the dynamic geometry environment Cabri.

Ruthven's "The instrumediation of mathematical activity and capability: Thoughts on instructional adaptation and learning facilitation" analysed the ways in which upper-primary-school students made use of calculators in tackling a division problem in 'using and applying mathematics'. Drawing analogies with recognized parallel strategies of written division, the presentation provided examples of a number of important phenomena: how a shift from written to calculator computation can augment student capability and sustain task strategy; how the low cost of calculator computation can encourage trialling of plausible variants, but effective use depends on understanding of procedures and results; how primitive cumulate-and-count strategies are relatively vulnerable to irretrievable error in the absence of recording, whereas trial-and-improvement strategies are more robust; how trial-and-improvement strategies involve formulating (direct or inverse) relationships between variables, prefiguring the idea of covariation;

how tailoring calculator technique insightfully provides scope for developing mathematical understanding; how developing effective use of a calculator is linked to developing understanding of relations between number/fraction systems and division forms, and between dual-unit systems of measures/money and the place-value system of decimal numeration; and how there is currently no standard system of calculator division, comparable to recognized standard techniques of written division, to provide a cognitively efficient, socially recognized, and mathematically theorized system of techniques.

Session 3. Teachers' cognition and students' formation of mathematical conceptions, notions, strategies and beliefs

In this session the presenters *Tânia Campos* and *Sandra Magina*, (Brazil) and *Pessia Tsamir* and *Dina Tirosh*, (Israel) focused on various aspects of teachers' cognition and discussed its role in mathematics learning and instruction.

Campos and *Magina*: "Teacher's conceptions of fractions and their teaching strategies". The aim of the research presented in this talk was to investigate Brazilian primary school teachers' concepts of fraction and their teaching strategies. The main hypothesis was that teachers will be able to solve fraction problems in these different situations, but will display a limited range of teaching strategies when proposing ways of helping children overcome misconceptions about fractions. It is possible that their own knowledge of the invariants of fractions remains implicit in situations, that they have not explored in a more systematic way. The methodology used was to ask 70 primary school teachers to answer a questionnaire containing items where they solved some fraction problems and reacted to answers that were presented as children's answers, which displayed misconceptions that have been observed in previous research. The study concluded that these teachers displayed adequate concepts of fraction in most of the situations, but the majority showed some confusion about representing situations numerically through fractions or ratios. As expected, their main teaching strategy was to use concrete materials or drawings to facilitate perceptual comparisons. In the ratio situation teachers could solve the problem through ratios but most did not make a connection between ratio and fraction part of their teaching strategy.

Tsamir and *Tirosh* in their talk "What types of content knowledge are needed to teach mathematics: The layers' model" focused on one main issue: The types of mathematical knowledge that elementary school teachers need for leading rich mathematical discussions in their classes. They introduced a three-layer model of the subject matter knowledge needed for teaching mathematics in these grades, comprising: the *Core Layer*, the *Wrapping Layer* and the *Meta Layer*. A segment of a discussion in a sixth grade on the divisibility of sums of natural numbers was used to illustrate the approach. In this case, the *Core Layer* encompasses the *arithmetical knowledge* needed for understanding the various, related statements and justifications (e.g., knowledge about natural and operations with natural numbers, even and odd numbers, prime numbers, factorization). This kind of knowledge is quite straightforward and obviously necessary for evaluating arithmetic statements. The *Wrapping Layer* consists of a*lgebraic knowledge* that may grant a teacher a powerful tool to reach the correct conclusions regarding the validity of arithmetic statements that are made by the students in their class in a swift and efficient manner. The *Meta Layer* is the *validation-refutation knowledge*, i.e. the knowledge of appropriate ways to prove (or refute) a given statement (e.g., What is considered a correct "general proof"? When is an example or a counter example sufficient for proving or refuting a given statement?). Tsamir and Tirosh discussed the potential implications of this model for elementary teachers' education and suggested possible extensions to other mathematical domains.

Session 4. Difficulties in learning mathematics

The first part of this session was devoted to two presentations. The first, by *Herbert Ginsburg*, (USA) described children's understanding of multiplication as reflected in their work with hand held computers. The second (*Terezinha Nunes*, *Peter Bryant*, and *Ursula Pretzlik*, UK) discussed the role of schema and working memory in primary school children's mathematical difficulties.

Ginsburg: "I didn't know they knew that! Using hand held computers to investigate children's understanding of multiplication". An essential step in helping children with learning problems is to gain an understanding of their thinking. Children experiencing learning difficulties do not merely lack knowledge. Rather, they fail because they use faulty concepts and buggy algorithms, but may at the same time possess interesting informal ideas and personal strategies that can serve as a foundation for productive learning. The goal was therefore to develop a Personal Digital Assistant (PDA) that could help teachers to acquire insight into student thinking as well as performance. The system, developed with Wireless Generation, helps teachers to use a simple form of clinical interviewing to assess several aspects of mathematics, from block play to algebra; learn about children's strengths as well as weaknesses; understand development over time; develop deeper theories of children's thinking; and learn to assess on their own, without the PDA. In the case of multiplication, the PDA helps teachers to investigate children's number facts, motivation and meta-cognition, mental calculation, concepts (models), writing, alignment and place value, and written calculation. At present the system is functional and Ginsburg's team is investigating how teachers use it and what they learn from it.

Nunes, *Bryant*, and *Pretzlik*: "The role of reasoning schemes and working memory in explaining primary school children's mathematics difficulties". Two approaches to the explanation of children's difficulties in mathematics are currently used in psychology. One approach, based on ideas related to how the brain works, suggests that children's mathematical difficulties can be explained by a reduced capacity to retain information in memory while operating on it. The second, based on constructivist theories in developmental psychology, suggests that mathematical difficulties are related to children's specific problems in using logico-mathematical reasoning schemas. Previous evidence for the working memory view is based on correlational studies that use performance in arithmetic tasks as the measure of mathematical ability. They described two longitudinal studies where the outcome measure of mathematical ability included problem-solving tasks that went beyond arithmetic. The predictive power of an assessment of working memory was compared to the predictive power of an assessment of children's mathematical reasoning. Both studies showed that working memory was not a significant predictor of mathematics achievement in school whereas the assessment of reasoning schemas was, even after controlling for the children's general intelligence and knowledge of arithmetic. This finding was viewed positively because it is possible to improve children's reasoning but there are no good methods for improving their working memory.

The second part of this session consisted of short presentations of eight selected papers. Three papers described and discussed the formation of mathematical concepts and ideas: fractions (*Susan B. Empson*, *Debra L. Junk*, *Higinio Dominguez*, *Kevin LoPresto*, and *Erin Turner*, USA), proportion (*Olof Bjorg Steinthorsdottir*, *Kristjana Skuladottir* and *Maria Sophusdottir*, USA and Iceland) and geometry (*Alexandra Gomes* and *Elfrida Ralha*, Portugal), and five papers described various aspects of students' and teachers' conceptions of mathematics (*Rita Borromeo Ferri*, Germany; *John Francisco* and *Carolyn Maher*. USA; *Ok-Ki Kang*, Korea; *Willy Mwakapenda*, South Africa, *Mihaela Singer*, and *Cristian Voica*, Romania).

This report has been written by Terezinha Nuñes and Dina Tirosh. Dina Tirosh will be happy to be contacted at dina@post.tau.ac.il for further information on the work of this TSG.