The development of mathematical reasoning: A sixteen-year study

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Regular Lecture

Introduction

Insight into the development of mathematical reasoning in students comes from a detailed analysis of data from an extensive collection of videotapes and written work from a sixteen-year longitudinal study. Through analyses of case studies in the thinking and reasoning of a cohort group of thirteen students, this paper traces the transformation of early ideas and images into mathematical symbols and the images that the symbols carried for the students. The research program began with an entire heterogenous class of first graders for three years, and continued with a subset of thirteen students through high school, and seven of the students through their college years. This report is based on students' investigations of a collection of problems from one of the problem strands, counting and combinatorics. An important finding from the study is that fundamental ideas and images, indicated by the students during their problem-solving activity as young children, were later elaborated, in symbolic expressions of generalized concepts. It was found that the students displayed deep and durable knowing about these concepts when interviewed as university students.

The invitation to follow one's own level and search for various paths to investigate mathematical ideas is appealing. Freudenthal reminds us that there is no single path in the building of mathematical ideas (1991). His advocacy to provide learners with opportunities to find their own levels and to explore multiple paths challenges us to find situations to make this happen. How might we provoke thinking? What might learners do? According to Freudenthal (1991):

> "the learner should reinvent mathematising rather than mathematics; abstracting rather than abstractions; schematizing rather than schemes; formalizing rather than formulas; algorithmising rather than algorithms; verbalizing rather than language" (p. 49).

Activity, thoughtfulness, and thinking have long been viewed as desirable human activities. Einstein (1936) referred to thinking as "operations with concepts". He described the thinking process as involving the creation and use of explicit functional relations between concepts and the coordination of sense experiences to these concepts. He wrote about the "awesomeness" of thinking and expressed skepticism about ever coming to understand how thinking actually works. Cognitive psychologists, cognitive scientists, and others interested in how learning occurs continue to study the process of thinking. Individuals and groups focus on different assumptions for their inquiry. The characterizations of processing systems, symbol manipulation paradigms, and the inferences made from observational and empirical research vary. The different frameworks for studying thinking and the assumptions that support these frameworks continue to be the subject of debate among theorists, policy makers and practitioners. Even so, inquiry into the origin and growth of mathematical ideas and about how learners reason about their ideas continues to be of interest.

A perspective on thinking and reasoning

If, indeed, it is the responsibility of the learner to build up a collection of mathematical ideas in his or her mind, it is reasonable to ask where those ideas come from. Elsewhere, we have argued that new ideas come from old ideas (Davis and Maher, 1997). Learners have ideas in their mind and can work creatively with them. When they come to their investigations, they bring extensive personal experience. This experience, along with their existing mental representations, knowledge, and beliefs, may be modified and refined, as needed, in contexts that include a mixture of personal exploration and social interaction. Given a task, learners decide what to investigate, how to go about an investigation, and what to examine. In the process of their investigation, they compose a discourse of their own, and from this discourse, working theories emerge. What about the potential of working theories of learners? These theories can result in effective ways of working with mathematical ideas. Through personal experience with concrete objects, or with images derived from working with these objects, specific arguments are built. When learners revisit their ideas from new perspectives, their arguments may serve as prototypes for later, more abstract reasoning. It is from learners' emerging theories and the way they work with them that an understanding of their work and thought can be built.

The research presented here is an observational study detailing the collective building by learners of mathematical ideas and ways they reason about the ideas. Thinking is inferred by the actions of the participants, that is, what they do, say, and write. Our extensive videotape data, provide a rich data base to study the complexity of the group's learning. From this data base, examples of ways of reasoning and of mathematical ideas that the cohort *built together* over the years will be given (Maher, 2002; Maher and Kiczek, 2000; Maher, C. A. and Martino, A. M., 1996; Maher and Speiser, 2000).

Certain views about learning and teaching, and what it means to do mathematics give rise to particular conditions for the study (Davis and Maher, 1990; Maher, 1988). Central to the learning environment is respect for individual choice, independence, and autonomy. Another important aspect is task design. In response to attentiveness to developing ideas of the learners engaged in the investigation, task review is central. What learners do and question may suggest revisions or the design of new tasks.

The research questions guiding this study evolved over the years. A major question at the onset was: How are mathematical ideas built by learners? A second objective, the development of student reasoning, emerged a few years into the research, following the observations of conversations of children whose arguments, naturally, took the form of mathematical proof. Another development was to give more attention to the researcher in interactions with children. We were intrigued by the phenomenon of students' *learning* when there was no intent for *explicit* teaching. Consequently, the following research questions emerged:

(1) How do mathematical ideas and modes of reasoning develop in learners?
(2) What is the role of the researcher/teacher in the process?

This paper will report on several episodes of the sixteen-year study to address those questions.

Background and motivation for the study

Prior to reform in mathematics education in the United States, mathematics classroom teaching, in general, and the mathematics instruction in the school district where the study took place, in particular, was essentially drill and practice. Students were expected to memorize facts, imitate procedures shown to them, and practice for mastery. The allocated time for classroom mathematical activity in most elementary schools was 30 to 40 minutes. In the project district where this study took place, a visionary principal sought to improve the situation in his K-8 elementary school, situated in the working class town of Kenilworth, New Jersey. He sought the assistance of this University researcher who responded, initially, by studying the existing situation: visiting and observing classrooms, interviewing teachers, and talking with administrators. The immediate outcome was an agreement for a 3-year school/university teacher development collaborative and tentative plan for monitoring student learning (Maher, 1991; Maher, Davis, and Alston, 1991). The partnership lasted over a decade and resulted in the longitudinal study that continued through the high-school graduation of its original first-grade participants (Maher, 2002). In 1988, a research project was designed to look at, carefully, students' building and constructing of mathematical knowledge and ways of knowing as they worked on well-defined strands of problems in several content domains over several years. A component of the contract for the university/school partnership included a plan to study the mathematical development of a class of first-grade students who stayed together for three years. In 1992, with support from the National Science Foundation (NSF), the research plan was expanded to include two other districts, one urban and the other, rural suburban. The cross-sectional and longitudinal studies were carried out in the three districts for four years. A second NSF grant extended the original Kenilworth study through the students' high-school years[1].

Design of the study

In the first eight years of the study, the research was classroom based. An entire first-grade class to which students were randomly assigned was the original focus group of the study. The students remained a class for three years, according to school policy at that time. Researchers introduced problem explorations for the students approximately six times a year. The task-based interventions lasted about 3 days, often with follow-up individual and/or small group interviews of students about their problem solving. The mathematics content for the tasks was not a part of the school curriculum for that grade. The interventions might be characterized as an infusion of thoughtful investigations; invitations to students to collaborate; sufficient time to work, and a didactical contract that left unresolved mathematical problems open to revisit at another time. An emphasis was to value explorations of the wealth of possibilities afforded by the conditions of the task.

Time allocation for doing mathematics. Allocated time for students to work on tasks was determined by student interest and engagement. The decision to revisit the same, similar or extensions of the problem was that of the researcher. The determination of the correctness and validity of proposed solutions was left to the students who were asked to provide a convincing argument. It was established, from the onset, that

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researchers would neither reject nor affirm the solutions posed. The responsibility for establishing validity of solutions was the responsibility of the students. The students were not graded for their work; we left it to them to decide on the criteria for successful completion. This produced a context in which students relied on each other to share and exchange ideas and explain their reasoning. They argued about alternatives and communicated clearly and effectively with each other.

Role of the researcher. The researcher who facilitates the session is referred to as teacher/researcher (T/R). Responsibilities of the T/R include: the design, selection, revision, and extension of problem tasks; posing of investigations; inviting student exploration and justification; attending to the developing ideas of learners; and the facilitation of sharing of ideas and results.

Expectations for students. Primary expectations for student participants were: to make judgments and strategic choices about what features are essential for the problem task; to invent, to experiment with, and to select representations, notations, terminology and modes of argument; to engage with the uncertainties and ambiguities of work in progress; to try ideas and share ideas and information; to become aware of others' work and thinking; to test conjectures, to pose questions to one's peers; to build lines of reasoning for oneself; to convince and/or question others; to develop and recognize proposed solutions; and to provide convincing arguments.

Methodology. The observational research uses videotape methodology for an extensive collection of over 3500 videotapes[2] that is described elsewhere in detail (Davis, Maher, and Martino, 1992; Powell, Francisco, and Maher, 2003). The data are analyzed in the context of investigations of the development of reasoning of students engaged in open-ended problem investigations in counting and combinatorics. The research was conducted in classrooms, in after-school sessions, in a summer institute, and in individual and small-group interviews. The analytical model for studying the development of reasoning involves attentively viewing videotape data; describing the video data in particular time intervals, flagging and then transcribing critical events, coding according to research questions, writing a storyline, and then composing the narrative. Researchers observe, describe, code the videotape data; keep written and electronic files of their emerging theoretic, analytic, and interpretative ideas about the students' behaviors, that is, their reasoning, use of inscriptions, connections between and among codes, and their emerging and extended ideas and ways of reasoning as depicted in e-mail transcriptions and revisited written and videotaped work.

Task design. Problem tasks[3] were designed to be well defined, open ended and challenging. They were intended to be accessible to a wide variety of students, who could be engaged and successful at some level. It was of interest to learn how students made sense of complex situations, what heuristics they used, and how they reasoned about the correctness of their solutions. Inherent in problem design was the potential

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[2]For journal articles and video episodes of students solving several of these problems, see the Private Universe Project in Mathematics, a Professional Development Workshop Series for K-12 Teachers of Mathematics produced by theHarvard-Smithsonian Center for Astrophysics in collaboration with the Robert B. Davis Institute for Learning, Rutgers University. Also, visit the web site: www.learner.org/channel/workshops/pupmath

[3] For detailed descriptions of problem tasks, visit the Rutgers University web site for the Robert B. Davis Institute for Learning: www.rbdil.gse.rutgers.edu

for extending and generalizing the problem. This provided an opportunity to trace the development of reasoning over time. It should be noted that the strand of problems posed to the students was deliberately designed to be more difficult than normally encountered for the age and grade level of the participants. It was expected that students work on problems for long periods of time. Our interest was in learning about how students' made sense out of complex situations, what heuristics they used, and how they modified, supported and extended their ideas over time.

An overview of counting strand[4]***.*** Several examples of tasks are given to illustrate the initial reasoning of students and how that reasoning was refined and extended over the years as they investigated problems in counting and combinatorics. Early investigations of patterns and symbolic representations began with counting problems in grade one; combinatorial problems dealing with making outfits from differently colored shirts and pants in grades 2 and 3; assembling place settings consisting of differently colored cups, bowls and plates, in grades 3; and by building block towers, by selecting plastic cubes of different colors in grades 3 through high-school. In the middle grades and through high school, students were engaged enumerating pizzas with available toppings (on whole and half pizzas), and different crust thickness. In grade 4 students' justifications took the form of proof (Maher, 1995; Martino, 1992; Martino, A. M. and Maher, C. A., 1991; Martino and Maher, 1999).In the middle and high-school years, students investigated binomial coefficients and combinations in relationship to the binomial expansion and the mapping of the binomial expansion to Pascal's triangle in more general situations (Maher and Speiser, 2001). Students' earlier work provided a foundation for their later thinking about combinations, polynomials, Pascal's triangle, block towers, pizzas, and, in high school, taxicab geometry, In later years, the students reformulated their earlier ideas and processes and they situated and extended them in broader contexts. These reformulations provided a foundation for a more detailed analysis of tower and pizza problems, and variations of them.

Results

Episode 1: Counting, grade 1. Four children were working as a group on the following problem:

> The kangaroo jumped six times. If the rabbit jumps four more times, he will have jumped as many times as the kangaroo. How many times has the rabbit already jumped?

One of the boys, Gerardo, answered: "So it would be six...because the kangaroo jumped six times. Stephanie responded: "Wait a second buddy...you just can't say six." Gerardo repeated his answer. Stephanie took plastic cubes and talked aloud as she worked out the solution. She said: "You just can't jump to conclusions like 'I know this.' Let's just try these five...no six...jumped six [organizing six unifix cubes as a group and removes four of them] six and four...two. Put two over here...one, two. We did it! It's two. Do you want to go over the problems and figure out if they're right?"

The episode indicates a natural, early interest in model building and sense making by first grader, Stephanie, to justify a solution to a problem (Martino, 1992; Martino, A. and Maher, C. A., 1991).

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[4]See websites for problem tasks.

Episode 2: Tower building, grades 3-6. During the early years of the study, students modeled solutions to counting problems with a variety of physical materials available to them. Their constructions were guided by specific observations. Students looked for processes that generated patterns. They reported on relationships between patterns; showed evidence of controlling for variables; invented and used notations. They devised and applied checking strategies such as exhaustive searches; they displayed a variety of representations of the models they built, including drawings and symbolic expressions that they invented.

In Figure 1 below, Stephanie refers to her solution of 16 towers for finding all possible 4-tall towers, selecting from cubes available in two colors, to justify her solution of 8 towers, for finding all possible 3-tall towers (Martino, 1992). In Episode 1, third graders Dana and Stephanie reported that they found 16 different combinations of 4-tall towers when selecting from plastic cubes of 2 colors. They displayed their towers and reported that they had eliminated duplicates by checking, exhaustively. While working together, they noted particular categories of towers and invented labels such as *staircase*, *elevator*, *opposite*, and *cousin* to make distinctions among groups of towers. They carried on a careful, exhaustive search to eliminate duplicate towers. They reported that they were satisfied that they found them all (Maher, Davis, and Alston, 1992; Maher and Martino, 1992; Maher & Martino, 2000).

Figure 1. Towers 4-tall to 3-tall, Dana and Stephanie

With an arrangement of 4-tall towers in front of them, the girls were asked by the researcher whether they expected the number of 3-tall towers to be fewer, the same, or more than the 16 four-tall towers displayed in front of them. They responded that there would be more, explaining that when they removed a cube from the top of each of the sixteen 4-tall towers, they would still have sixteen towers, now 3-tall, plus 16 additional plastic cubes to build even more 3-tall towers. The researcher then asked them to illustrate this with the cubes. In their attempt to do so, they noticed duplicate towers and reported that they had changed their mind. They reported that there were only 8 unique towers, and illustrated the emergence of duplicates. Their explanation indicated an understanding of reversibility in the process of tower building, that is, they offered an argument moving from n to (n-1) tall towers, when n = 4. The third graders built their towers, at first, using random, guess and check methods. They were asked as fourth graders to build all possible towers 5-tall, selecting from 2 colors, and justify that they had them all. The fourth graders were more systematic in their tower building and organized groups of towers according to patterns that eventually evolved to organizations by cases (Maher, C. A. and Martino, A. M., 1996).

In a video segment, of the Gang of 4, Jeff, Michelle, Milin and Stephanie collaborate on the challenge to convince each other and the researcher that they found all towers, 3-tall, selecting from two colors. In the video episode, Stephanie makes reference to towers by using a letter to represent the color of the cube, arranged by cases; Milin draws pictures of towers, showing how they grow taller by attaching each color to a tower of given height. The students work with images of towers, now represented as drawings and symbols, rather than the physical tower objects. They notice certain properties about these images and suggest an organization for categorizing the images into complete and exhaustive classification schemes. These schemes are the precursors to the argument by cases and contradiction provided by Stephanie and the inductive reasoning argument introduced by Milin. They give evidence of the children's early understanding of the idea of proof. Figure 2 shows Stephanie's solution by cases for accounting for all possible towers, 3 tall, selecting from two colors (Maher, C. A. and Martino, A.M., 1996).

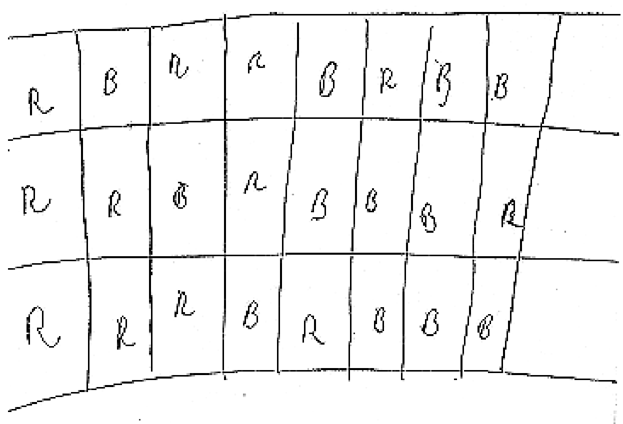

Figure 2. Stephanie's solution for 3-tall towers

The children produced expositions of their own and then shared their individual interpretations of their reasoning. In response to questioning, they offered warrants for their reasoning, sometimes making adjustments that reflected the input from others. Through a process of sharing and revisiting earlier work, they participated in a critical review of each other's ideas which, in turn, led to a more thorough reconsideration of earlier held ideas. Their classification schemes for organizing towers expanded from local to global, and to the invention of the idea of proof.

Episode 3: Pizza problems, grades 5-11. In grade 5, the students were introduced to a strand of problems involving finding the choices for ordering pizza, given a variety of toppings available. The initial pizza problem posed was:

> *The Pizza Problem.* Pizza Hut ® has asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from four toppings: peppers,

> sausage, mushrooms and pepperoni. How many choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possibilities.

Working together as fifth graders (4/2/93), the students applied several strategies. These include making a partial tree diagram, creating lists, and an organization that systematically controlled for variables. Michael worked in parallel with the others and represented the various pizzas by drawing circles and labeled each "pizza" with a topping choice. The students agreed that there were sixteen pizzas choices and expressed jubilance with their results (Figure 3).

Figure 3. Michelle, Jeff, Romina and Ankur find solution

When revisiting the problems over four years later in grade 10 (12/12/97), the students reported that if five toppings were available, thirty different pizzas could be made with at least one topping, plus one plain cheese pizza, for a total of thirty-one. Michael argued there were thirty-two pizza choices and challenged the solution of the others. To justify his result, he used a representation based on a binary-coding scheme to account for all possibilities and matched the four toppings to the four places in the binary number. He wrote:

> "You've got four toppings. This is like four places of the binary system. It all equals up to fifteen. That's the answer for four toppings." He continued to explain: "Okay, here's what I think. You know like a binary system we learned a while ago? Like with the ones and zeros - binary, right? The ones would mean a topping; zero means no topping. So if you had a four-topping pizza, you have four different places - in the binary system, you'd have - the first one would be just one. The second one would be that [He wrote 1 0]; that's the second number up. You remember what that was? This was like two, and this was three [He wrote 1 1]….Well, you get, I think—… You've got four toppings. This is like four places of the binary system. It all equals up to fifteen. That's the answer for four toppings. So you go from this number [He pointed to 0 0 0 1], which is in the binary system, it's one, to this number [He pointed

> to 1 1 1 1], which is fifteen, and that's how many toppings you have. There's fifteen different combinations of ones and zeros if you have four different places...and if you have an extra topping, you just add an extra place and that would be sixteen, that would be thirty-one."

In a session, one year later (Grade 11, 12/14/98), Michael explained his binary representation with respect to pizzas with three toppings, referring. to the list of binary numbers he used to code pizza choices (Figure 4). He indicated 000 as the code for "no toppings", and 111 for a pizza with "everything on it." Referring to the rows of Pascal's triangle, he demonstrated the addition rule by adding more toppings in making pizzas.

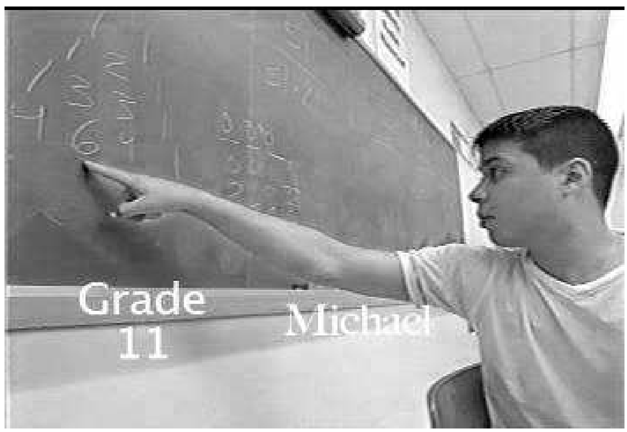

Figure 4. Michael explains the addition rule with pizza examples

In a subsequent e-mail message (January, 1999) Michael provided a written justifycation for why the numbers in Pascal's triangle mapped to the variety of choices for making pizzas, and illustrated this by moving from pizzas with four toppings to pizzas with five toppings (Kiczek, Maher, and Speiser, 2001). He made reference to four toppings and mapped a four digit binary number in correspondence to the 16 pizza choices and indicated that there was a connection to Pascal's triangle. He drew Pascal's triangle and referred to 1 when a topping was present and 0 when a topping was not present. He represented (0) with (0000) for no toppings and listed consecutively binary numbers to 15 with (1111) for all toppings. He made a list, placing the binary numbers in 1-1 correspondence with the topping choices. He then referred to rows 4 and 5 of Pascal's triangle. Michael wrote:

> "You see, in the pizza example above there were
>
> - 1 pizza without toppings
> - 4 with one topping
> - 6 with two toppings
> - 4 with three toppings
> - 1 with all four.

How come these numbers look so familiar???Look at the fifth row of Pascal's triangle! If you added another topping to the list like anchovies, you would have:

- 1 with 0 toppings
- 5 with 1 toppings
- 10 with 2 toppings
- 10 with 3 toppings
- 5 with 4 toppings
- 1 with 5 toppings

Why? Here's why. The way you make the triangle is by taking the numbers from the row up ahead and adding every two together.(think of having zeros on each side of the "triangle").

1 3 3 1 -> 0+1, 1+3, 3+3, 3+1, 1+0

1 4 6 4 1

In the 1 3 3 1 sequence the first position represents the binary numbers with all zeros (no topping pizza) but when you add another topping, it could either have it or not:

0000
/
000< or
\\
0001

So now this 1 pizza combination turned into two. So do all the other combinations. When you add a topping you put a one on the end of it if not you put another zero onto it. This explains why the Pascal's triangle works:

We add because of the fact that every combination will get another place (a 1 and a 0) therefore it doubles the amount of combinations. The reasons why we add numbers that are next to each other is simply because the underlined 1 will become into two new pizzas after another topping choice is added. One of those will be the same (no toppings) and the other will have one topping. The same will happen for the 2. The two will become 4. 2 the same (one topping) and two with two toppings. Now you have three one topping pizzas. 1 comes from the 1 in the "upper level" of the triangle, and two come from 2 in the "upper level". That is why ya add 'em… e-mail me soon with your response."

Episode 4: Revisiting towers, grade 10. In March, 2002, Ankur, Brian, Jeff, Michael and Romina revisited tower problems (Muter, 1999; Muter and Maher, 1998). They formed two groups: Michael and Ankur; Brian, Jeff and Romina. They were asked to provide a convincing argument for finding all possible towers, selecting from 2 colors, 5-tall. Michael and Ankur used binary numbers and quickly solved the problem using an exhaustive argument. While the other group worked to provide an argument by cases, Ankur posed another problem, *Ankur's Challenge*:

> How many combinations can you make with towers four tall, selecting from a choice of 3 colors and using at least one of each color in every tower?

Two solutions emerged, one from each group (Maher, 2002). Ankur and Michael first calculated the number of 4-tall towers, selecting from 3 colors and then subtracted out duplicate sets, double counting 3 towers. Brian, Jeff and Romina partitioned the set of all possible towers into six groups. They indicated that every tower would have two of one color; they indicated the placement of the duplicate color using X and O. They accounted for the placement of the duplicate color, indicating that for each placement of the first duplicate color, there would be two possible choices for the second and third colors, and tripled the 12 outcomes to represent every color, arguing that there would be 36 possibilities (Figure 5). Romina, then, presented their solution to the others (Figure 6).

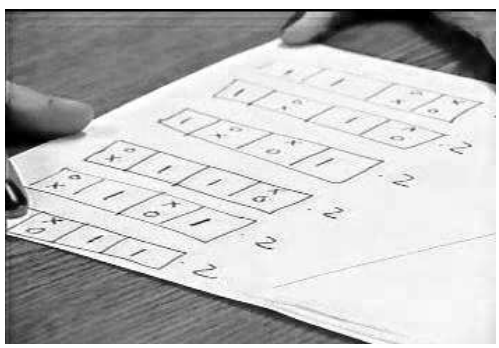

Figure 5. Students, grade 10, solve Ankur's challenge

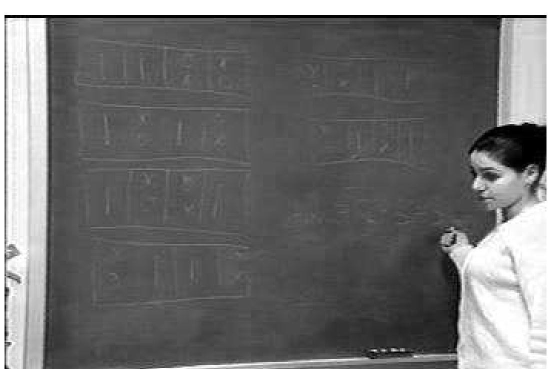

Figure 6. Romina's presentation of her group's solution

Episode 5: Grade 11, Night Session. Five students gathered late on a Thursday evening, May 12, 1999 for a problem solving session. The session is notable because it was this evening that they formalized their understanding of Pascal’s identity using standard, symbolic notation (Maher, 2002; Uptegrove and Maher, 2004b). Referring to the actions of building towers and adding pizza toppings, and using binary notation introduced by Michael during earlier sessions, they explained, together, the general addition rule (Figure 7).

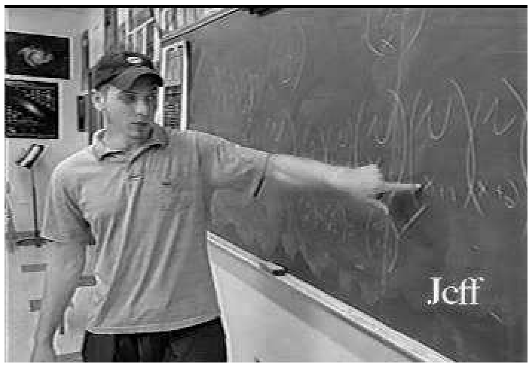

Figure 7. Jeff uses pizza metaphor to explain addition rule

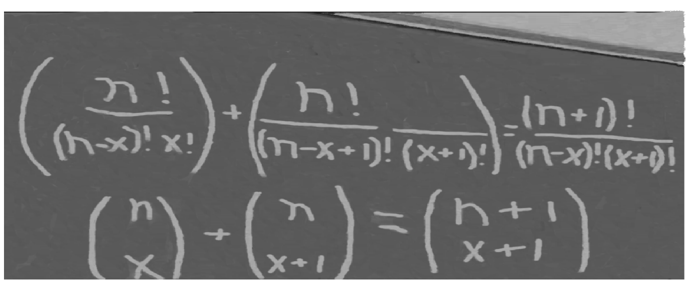

Figure 8. Students express addition rule with factorial notation

Episode 6: Taxicab problem, grade 12. In senior-high school and again in early university years, the students continued to explore the meaning of Pascal’s addition rule. The culminating high-school combinatorics task was the Taxicab Problem, in which students are presented with a Cartesian grid representing the specific territory for a town and on which particular intersections are designated. The taxicab driver is dispatched only three times to pick up passengers at one of the intersection on the map. She considers all possible routes to determine if there was a shortest. Students are asked to investigate possible routes to advise the driver. Through an analysis of Pascal’s identity, the students illustrated a three-way isomorphism pointing out the similarity in structure of the pizza, towers, and taxicab problems. These became metaphors that were interchanged by students when working on the problem (Powell, 2003; Powell and Maher, 2003).

Episode 7: Reflecting on learning, grades 11-university. When asked to reflect on their learning of the mathematics, the students emphasized the importance of personal knowledge, collaboration, arguing, justifying and convincing themselves and each other about the validity of their reasoning (Francisco, 2004).

Romina, a University of Pennsylvania student comments on the way they worked:

> "If I didn't understand the problem, or if I didn't work enough to it, by myself to understand, and I guess if Michael didn't know where I was heading with what I was doing, and if I didn't understand where the other person was heading I would like to work on it before I came up with a couple of options myself to see which one we take. " (Romina, (March 2002)

Jeff, as an 11th grader notes:

> "Well, even though we didn't spend much time together, and they [researchers] only came a few times a year, we did so much, we covered so much, and we got so in-depth on topics, that it leaves an impression. I mean, we could talk about doing the blocks in first grade, and we can almost go through problems: We did shirts and pants in second grade. I mean, how many other people can tell you the math that they were doing in second grade…like a word problem, you know? Because you go in deep, you work on it so much, and you go so far into it, that it just sticks with you…That's why it leaves such an impression, because of the depth you get into it." (Jeff, May 1999)

Three years later, as a Rutgers University student, he remembers:

> "I think it would have been very different if it was all of us producing our own solutions…. I think a lot of what we were doing was working together. I think when you are working alone, when you reach a part where you don't know anymore it is very easy to just be frustrated and say I don't know anymore. I'm not going to do this. I can't think about this. Like forget it. I think that by working with everybody when you got to that point, you can kind of peak over a little bit and it was all right...it was encouraged. That allowed everybody to really… we could all move forward." (Jeff, March 2002)

At the end of their high school year and during university years, several students continued to participate in individual and small group interviews about their earlier mathematical activity. Figure 9 shows Brian and Romina revisiting earlier problems from the combinatorial strand. During these interviews, they re-constructed explanat-ions of Pascal's Identity, using both general notations and the images they had of towers, pizzas, binary numbers, and taxicab routes . Over the years, they identified the structural identity between problem situations that were not on the surface readily apparent. Their constructions were detailed and thoughtful; their building of three-way isomorphisms, is desribed in recent work and work by Uptegrove in progress. (Powell, 2003; Uptegrove and Maher, 2004a).

Figure 9. Romina and Brian revisit problems

Conclusions

The study, now entering its seventeenth year, has focused on the development of mathematical ideas and reasoning in learners. The students were invited to think about mathematical situations, often over long periods of time, and to present their ideas with justifications that were convincing to them and to us. They were invited to consider generalizations and extensions of these situations. In the process of documenting the long-term development of their mathematical reasoning, the data revealed an important and interesting interplay between ideas brought forward by students and the conditions under which those ideas were examined. An important finding is the co-construction of knowledge in student-to-student interactions. Both the cognitive aspects and the social and cultural milieu shaped the building of arguments. There is an interesting balance between the strong ideas brought forward by individual students and the social process in which the ideas, as they traveled through the community, were valued as contributions, openly discussed, evaluated, modified, extended, and justified. Particular key ideas built by students across three periods over the years are presented to illustrate the development of student reasoning.

In students' work on these and other tasks, we see certain patterns of reasoning emerge in the very early years of the study and continue throughout. Third and fourth graders look for patterns, guess-and-check, refer to a simpler problem, and systematically control for variables, that is, they hold fixed a component of a given organization of towers while letting other components vary. For example, they held fixed the number of blocks in a tower for a given color; the top block of a tower; the particular separation of a pair of blocks with the same color; and the position of the lowest block of a certain color. These strategies led to re-organizations of towers for arguments taking the form of cases, induction, recursion, and contradiction. Over the years, there surfaced an increasingly sophisticated use of metaphor for mapping particular ideas between related problems. Their observation of patterns in inscriptions for notations (e.g, binary numbers) and their detection of patterns in Pascal's triangle provided, yet, another way for sharing their understanding. The study provides insight into students' growing understanding of particular mathematical ideas, through their communal problem solving of a strand of problems. They exhibited creative thinking in their pursuit of sense making and justification in contexts that by design avoid scaffolding or prompts. They generalized beyond particular problem situations in their efforts to build convincing arguments.

Reflections by students about their mathematical activity and how they structured their own learning, provided insight into how the students viewed learning mathematics together. Participants indicated that they were aware of what would be asked of them and took upon themselves the responsibility of developing satisfactory solutions. Videotape data document the process of student growth in reasoning and episodes from the extensive data base provide evidence of students actively engaged in doing mathematics across the years. They built, often together, a powerful repertoire of mental images to draw upon and use in their problem solving. These examples highlight their mathematical reasoning and engagement from early elementary years, through middle and high school, and during undergraduate university years. In the early and middle years, the work of students was displayed as images of objects. They invented the idea of proof and used forms of proof to structure their arguments. In their later years, they displayed images of schemes and relationships between and among schemes. In later years, students displayed higher level structures (isomorphisms) by mentally rearranging earlier forms of images and actions on those images. Inferences that students drew from their early images (such as arrangements to make outfits, sets of towers, combinations of pizzas, and various taxicab routes), played an important role in their later understanding about general and new combinations, expressed, first, by notations invented by them. Hence ideas and processes from students' activity in their earlier engagement with these ideas, were later retrieved, sometimes modified and reformulated, and often extended as they reconstructed new, more abstract, and general ideas.

In follow up interviews, the students reported that they maintained high expectations for themselves and for each other. They took on, progressively, greater responsibility for their own learning and for the maintenance of communication and collaboration with each other. Their confidence in their problem solving ability increased and this was evident in their willingness to tackle new and challenging problems throughout the years.

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