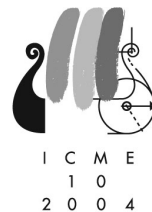


Exploiting the gap between intuitive and formal knowledge in mathematics

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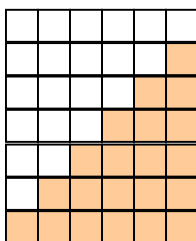


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Introduction

A mathematical task often treated in educational contexts is about how to evaluate the sum of the first n natural numbers, i.e. to find a closed form expression of $s_n = 1 + 2 + 3 + \dots + (n-1) + n$, for any natural number n . A common approach is to supply a diagram (with origin in the ancient history of triangular numbers), from which the result may be directly “observed”:



By using this diagram¹, it is possible not only to find the general formula but also to get a feeling of understanding of why it looks the way it does, i.e. $s_n = \frac{n \cdot (n+1)}{2}$.

Now, the same sum $s_n = \sum_{k=1}^n k$ may also be computed by the following (standard) method:

$$(n+1)^2 - 1 = \sum_{k=1}^n ((k+1)^2 - k^2) = \sum_{k=1}^n (2k+1) = 2 \sum_{k=1}^n k + n, \text{ i.e. } s_n = \frac{1}{2} n^2 + \frac{1}{2} n$$

Since the result is produced by a logical sequence of (correct) steps, one is convinced about its validity. On the other hand, the computations do not “reveal”, in a manner as obvious as the diagrammatic solution above, why the formula looks the way it does. To learners, this formal solution may seem not only somewhat complicated but also not to provide any insight into the structure of the problem. So why bother a learner with it? What does such “purely formal” mathematics add to the more appealing intuitively based insights? In this case, it introduces a method by which this *type of problem* may be solved, where the intuitive approach is insufficient – to find the sums of the squares, the cubes, and so on iteratively. For example, with $u_n = \sum_{k=1}^n k^2$ the computation above is easily “copied”:

¹ This may also be done, of course, only numerically by adding both forwards and backwards, without reference to a diagram.

$$(n+1)^3 - 1 = \sum_{k=1}^n ((k+1)^3 - k^3) = \sum_{k=1}^n (3k^2 + 3k + 1) = 3 \sum_{k=1}^n k^2 + 3 \sum_{k=1}^n k + n,$$

i.e. $u_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$

The example serves to illustrate how an intuitive world and a formal world enter into mathematical work, with their different impacts on understanding and mathematical production. How these worlds are related has been an ongoing discussion in mathematics education and will in the following be my focus of attention. Being facets of human endeavour, both worlds are situated in the “sea of experiences” at the bottom of which some hidden connections may be found. Is there a gap between intuitive and formal knowledge in mathematics, if so, what is the nature of this gap, and how can it be exploited for the benefit of learning mathematics?

Intuitive knowledge in mathematics

The critical role of intuition in doing and understanding mathematics has long been emphasised, though more by mathematicians and philosophers than by mathematics educators. Kitcher wrote, for example, that “‘Intuition’ is one of the most overworked terms in the philosophy of mathematics” (Kitcher, 1984, p. 49), which did not prevent him from giving it a whole chapter in his book. The intuitionists, such as Brouwer and Heyting, placed the real mathematics in the mind and not as signs on a paper but also Hilbert and Poincaré are frequently quoted on emphasising the key role of intuition in mathematics.

In the context of mathematics education, Efraim Fischbein’s pioneering work on this topic is well known. He talked on intuition and mathematics teaching already at ICME 2 in Exeter (Fischbein, 1973). He defines intuitive cognition as “a kind of cognition that is accepted directly without the feeling that any kind of justification is required” (Fischbein, 1994, p. 232) but also states that there “is no commonly accepted definition for intuitive knowledge” (Fischbein, 1999, p. 12). Historically, different views of the role of intuition include it as a (cf. Fischbein, 1987)

- a reliable source of true knowledge (as opposed to often misleading sense impressions); *Descartes*
- a method, i.e. a mental strategy to search the essence of phenomena, beyond logical argumentation; *Bergson*;
- the faculty through which objects are directly grasped (to be distinguished from the faculty of understanding for conceptual knowledge), i.e. sensual intuition, often referred to in pedagogy; *Kant*
- a category of cognitions, directly grasped without a need for justification or interpretation; *Piaget*.

Intuition is often linked to common sense, as opposed to scientific knowledge, sometimes even to be avoided in science. Others claim that intuition is necessary for creative activity in science (e.g. Poincaré). The French psychologist Reuchlin separated *natural thinking* from *formal reasoning*, where the former has an adaptive function, characterised by “immediacy, concreteness, and capacity for sudden and global evaluations” (Fischbein, 1987, p. 5). These aspects are included in Fischbein’s view of intuition, which he sees as intuitive knowledge, i.e. a type of cognitions (in line with Piaget’s view) which are characterised by (1987, p. 200-201)

- self-evidence and immediacy
- intrinsic certainty
- perseverance
- coerciveness
- theory status
- extrapolativeness
- globality
- implicitness
- the cognitive-behavioral function of intuitions.

To Fischbein (1999), the evolution and functioning of intuitions are explained psychologically by mental schema theory. He elaborated especially on "the interaction between three basic components of *mathematics as a human activity*: mathematics, i.e. the formal, the algorithmic, and the intuitive", and emphasised the interplay between these as critical for doing mathematics (Fischbein, 1994, p. 231). The key role of intuition for mathematical thinking was discussed by Michael Otte at ICME 7 in Quebec, where he claimed that mathematics is the "incarnation of intuitive knowledge" (Otte, 1994, p. 272). In the philosophy of embodied cognition, with its elaborations in mathematics by Lakoff and Nuñez (2000), mathematical concepts (at least at the elementary level) grow out from bodily based image schemata by metaphorical projections, mainly unconscious processes thus producing intuitive feelings of recognition and understanding. Tall (2004) describes cognitive growth in mathematics as a journey through three interacting worlds, the embodied, the proceptual, and the formal world (roughly corresponding to the basic human triptych of perception, action, and reflection, respectively). In each of these worlds statements are validated by different means, intuition being seen as the initial 'warrant for truth' in the embodied world.

Formal knowledge in mathematics

The word formalism, when discussing mathematics, normally refers to either the symbolic language used specifically in mathematics, or to the logic and rigor of the exposition. According to Fischbein (1994), "Axioms, definitions, theorems, and proofs have to penetrate as active components in the reasoning process. They have to be invented or learned, organized, checked, and used actively by the student." (p. 232). Now, rigor is most often communicated through the specific symbolic language of mathematics, making it difficult to separate these two aspects of formalism. This links the algorithmic component (as Fischbein calls it), i.e. the skill to perform solving procedures, to formal knowledge. The same view is included in the following definition of formal understanding in mathematics, complementing the idea of intuitive understanding:

"Formal understanding is the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning" (Byers & Herscovics, 1977),

criticising Skemp's (1979, 1982) distinction between symbolic and logical understanding.



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Of relevance here is that the specific formal (symbolic) language of mathematics, the algebraic code, has a double life: as signifier (i.e. a vehicle for meaning), and as signified (here a collection of objects that can be manipulated according to specific rules), to use the terminology of Peirce. The function of the algebraic code thus has a *structural* as well as an *operational* aspect (Bergsten, 1990). As a basis for diagrammatic reasoning (in the sense of Peirce; see e.g. Dörfler, 2004) based on this algebraic code, I defined the notion of *mathematical form* as typographical units (such as 2 , x , $\sqrt{\quad}$, \int , and so on), called *atomic forms*, or as patterns of atomic forms (such as III, the pattern of for example the numeral 825, or IOI, the pattern of for example $2+3$, or $2x+3=11$, where the pattern IOI of the left-hand side of the equation is *chunked* into I), called *molecular forms* (Bergsten, 1990, 1999). To operate on such kinds of (alpha-numeric) symbolic expressions of mathematics, an awareness of the mathematical forms, from both the structural and the operational aspect, is critical. By the schematic structures of these forms, also intuition comes into play.

Relations between intuitive and formal knowledge

Most often different kinds of problems related to an observed gap between intuitive and formal understanding in mathematics are emphasised (see e.g. Fischbein, 1994), thus using a defect oriented approach, but more seldom the didactical power hidden in the tensions caused by such a gap is discussed. By using formalism the limits of intuition may be transcended, such as making the problem driven step from potential to actual infinity possible. Formally defined non-intuitive concepts replace more concrete ideas, and by their existence as concepts they may (over time) acquire a more intuitive status (Fischbein, 1987). A confidence in the power of formalism may thus contribute to the development of intuition. There is an inherent power in naming (cf. Sfard, 2000). A reflection that cannot be avoided, by the way, is that the same may apply to the word *intuition*. By the act of putting a label, a name, to a cognitive phenomenon otherwise difficult to account for, or explain, it is being born as a concept and brought into existence. After it has been used it belongs to experience, the very basis for the development of intuitions.

Historically, the gap between intuition and formalism has probably been one of the 'driving forces', by the tension it has created, to overcome the limitations set by intuition. As an example, to the ancient Greek sonly the natural numbers were 'natural', intuitive. By comparing them it was possible to talk about $2/3$ as the relation between 2 and 3, but that didn't make $2/3$ a 'number'. It took a big 'jump' to overcome intuition and consider these comparisons as numbers in themselves (i.e. the rational numbers) and operate on them as if they were 'numbers', against intuition. The same thing could be said (later) about negative numbers, the irrationals, the complex numbers, as well as about operations themselves. What is at first considered a formal idea may thus develop to later become an intuitive conception.

In line with this, the distinction by Reuchlin mentioned above is echoed in the mathematics education context in thinking styles termed *natural* versus *alien* learners:

“A 'natural' learner always attempts to 'make sense' of experiences by connecting them to existing mental structures, looking for explanations and reasons based on those connections. An 'alien' learner, on the other hand, is willing to accept new experiences in an 'alien' way, building up isolated structures which deal with just those experiences”. (Simpson, 1995, p. 42)

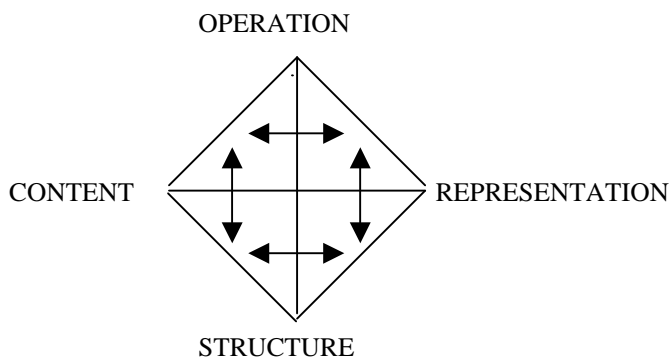


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In formalism there is an obvious need of symbol sense (see e.g. Arcavi, 1994; Bergsten, 1999), or a more general *diagram sense*, viewing algebraic transformation as one instance of diagrammatic reasoning in the sense of Peirce. What is formal, what is intuitive, in symbol sense? One must keep control at two levels - the global: ‘what do I want to do with this expression, what is its role in what is going on here?’, and the local: ‘what do I know about this expression and its parts?’ This double control may be supported by links to a more intuitive grasp of the ideas or concepts, and operations on these. These relationships were described in Bergsten (1990, p. 177) by the following diagram (called the ‘crystal’):



The arrows indicate mathematical activity. As an example, the equation $4x = x + 6$ displays a mathematical form (a link with a link as its left wing), by a person perceived through the lower right field in the crystal, with possible links to the left (intuitive) field by an interpretative act (lower arrow). Doing the symbol transformation of subtracting an x on both sides is, in the crystal, placed in the upper right field. The arrow on the right side also gives back the new form of $3x = 6$. The equation could also be solved intuitively, with no symbol manipulation, indicated by the arrows to the left. I thus defined *mathematical operativity* as an

“ability, in mathematical problem solving, to intentionally integrate conceptions of the mathematical content with the mathematical forms of the representing symbolic notations”. (Bergsten, 1990, p. 178)

Empirical data from lower secondary students indicated that this integration is mediated by logical thinking. Mathematical operativity thus creates a complete dynamics within the crystal, a kind of idealised model for the relation between some aspects of intuitive and formal knowledge. By interacting with intuition this way, the handling of mathematical formulas may develop to a symbol sense that will guide productive symbol transformations.

Even without a formal proof that a procedure works, it may work. The function of the proof is then to take away any doubts, to secure a trust in the result when using the procedure. The function of intuition is to give a sense of understanding, the function of the formal is to give a sense of security. The latter, however, takes a trust in formal mathematics as a method to provide such a stability or security of the deduced results. Such a trust may be either authority based, or established by experience.



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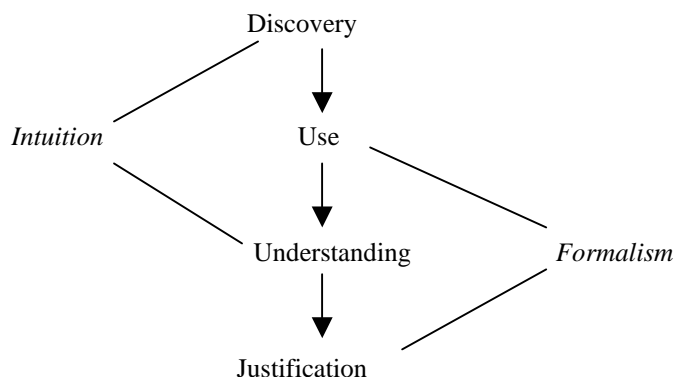
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An interesting example is provided by Newton's method for finding a zero of a function, a procedure where the relation between intuitive and formal understanding is far from simple. The intuitive appeal is immediate, already at the upper secondary level in Sweden for science students, and the method is easily applied to standard functions by the use of graphic calculators. A graphic outline supports the basis for elaborating a formula, displaying a nice bridge from the intuitive to the algorithmic aspect of formal knowledge. Any need for justification is seldom expressed by the students, and in most cases not found in the textbook. But the formula derived, i.e. the recursively defined sequence $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, is far from being as intuitively

obvious as the procedure itself, as shown by the graphics. And it is the formula that provides the result, not the graph. This is a strong case for the need of formal knowledge, based on intuitive reasoning from a diagram. In addition, the diagram does not give any justification – is the formula to be trusted, what should be required from the function for us to be sure that the procedure works, and so forth? Here an obvious and strong need for formal reasoning may motivate the student to elaborate on the (non-trivial) calculations, and lead into other intuitively based formal notions such as convexity.

Working with this problem may illustrate on an individual basis the four stages *discovery, use, understanding, and justification* that Kleiner (2001) identifies in the historical development of mathematical ideas. Intuition being most critical for the first and third stage, and formalism for the second and fourth, the figure below displays the intricate interaction of these two dimensions of mathematical activity.



What is the nature of the relation between intuitive and formal knowledge, as identified so far? The most critical divide between the two sides concerns the warrants for truth, and the ways to achieve it. On the intuitive side the key word to get certitude is insight, which is achieved by a direct intuitive grasp. On the formal side truth, or certitude, is achieved in an indirect way by reasoning. The 'code' is in the former case characterised by simultaneity, in the second case by a sequential format.

The didactic problem

Seeing mathematics as a formalisation of the organisation of ideas based on bodily experience, or situations in different disciplines (including mathematics itself), shaping the properties of the formalism (cf. Mac Lane, 1986), there is a genetic link



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between the form and content of mathematics, or between the formal and intuitive aspects of mathematical activity. This link touches upon both the operational and the structural aspect of mathematical thinking. Elementary mathematics being *content dominated* (and content generated), more advanced mathematics becomes *form dominated* (and form generated), at least on the surface level (Bergsten, 1990). In this context, the role of intuition in learning mathematics exhibits a Janus face - in elementary mathematics it is the key source for understanding, in more advanced mathematics it often creates epistemological obstacles – one has to learn when to trust it and when to ignore it. This puts the mathematics teacher in a dilemma, trying to find the way between the obvious and the rigorous, i.e. between the intuitive and the formal, in her attempts to guide her students to viable mathematical experiences. To be able to do this, she is constantly trying to come to grips with what she is teaching, elaborating on models of fractions or theorems on angle bisectors, within the world of school mathematics. Such efforts by teachers may be too demanding in the forced complexity of everyday practice, resulting in a split teaching focussed on intuitive empiricism or mechanistic training of formal procedures. *Consequently a gap between informal and formal knowledge may result as an outcome of this didactic practice.* Formalism is being stripped off its explanatory and inventive power and reduced to an algorithmic tool. As soon as the mathematics student is asked to explain a solution, this gap may open up: in addition to the student's intuitive image of the solution, discursive demands on presentation are forced upon the explanation. It has been argued that such aspects of formal knowledge as demands on rigour should not be a concern of first priority in school mathematics, as in this often quoted passage:

“The real problem which confronts mathematics teaching is not that of rigour, but the problem of the development of ‘meaning’, of the ‘existence’ of mathe-matical objects.” (Thom, 1973, p. 202; italics in original)

I do not believe one of these problems can be worked on without profiting from taking the other into account. The intuitive and the formal always go together, whether separated or linked. The didactic problem identified above is not resolved by hiding one of its parts, but both need to be objectified. Meaning builds heavily on representation, or rather representations.

Some empirical studies

Raman (2001) studied the coordination of informal and formal aspects of mathematics.

A college student intuitively claimed that the function $f(x) = \begin{cases} 0, & x \leq 0 \\ x, & x > 0 \end{cases}$, is

discontinuous at 0, because he “remembered” that a function with a pointy graph is discontinuous. Not being sure, when checking with the formal definition he found that it fulfilled the requirements for continuity, but he was still not satisfied:

“That pointy thing still sticks out of my head. But I don’t know why. Because its...yeah... it doesn’t jive with my intuition, but if I go back to the definition, yes, it should work.” (Raman, 2001, p. 5)

A pair of students (in the same study) were given the following problem in a situation out of context of a course (even though it was in fact taken from the textbook): *Is there a number that is exactly one more than its cube?* Writing down the correct equation, the students try some numbers, locating a root somewhere between -1 and -2 , trying

to find the exact root, using the derivative. After 50 minutes the teacher intervenes and restates the question in line with the intermediate value theorem questions in the textbook, and one of the students immediately wrote down a correct formal solution, responding when asked why it was difficult before to find the solution:

“Had you told us to open our books to page 89 and solve problem number 57, we would have done it in five minutes. Because it is, you know, after this section. So we know what we are supposed to do. But this just given like that, we don’t know, you know, which part of our knowledge to access.” (Raman, 2001, p. 15)

This student did not seem to be able to build on the informal understanding shown at the beginning of the session (root somewhere between -1 and -2). The following tendencies of student behaviour were observed in the study, where also a problem involving graphs was used:

- tendency not to connect informal characterisations of concepts with formal definitions
- tendency not to connect informal sense-making with theorem application (especially when the question is asked out of context)
- tendency to read too much into graphs, but also
- a reluctance to reason with a graph

The informal and the formal sometimes constitute in students two different worlds that do not conflict. Raman refers to explicit and implicit textbook messages to account for these student behaviours. Another conclusion is that students

“are given too little experience coordinating informal and formal aspects of mathematics for them to develop the tendency to do so without prompting.” (ibid., p. 18)

Small group activity sessions set up by myself for graduate students confirmed this out-of-context tendency observed by Raman. The (classical) problem, given to the students only for these particular sessions not related to any on-going mathematics course, concerned temperature distribution on the surface of a cylindrical wire, at a circular cross section: *Are there two diametrically opposite points with equal temperature?* There was no immediate or ‘natural’ attempt to provide a formal mathematical model of the situation. There was also no obvious hypothesis made concerning the answer to the question. The discussion between the students touched more upon concrete features of the practical circumstances, such as the material of the surface of the wire, though some mathematical issues, like the tangent function, were discussed. Even when the idea of a function to describe the temperature at different points of the circle was opened up, the step to consider a continuous function on an interval (of length equal to the circle perimeter) was not taken by the students, a step necessary to consider the application of the intermediate value theorem. Only after a second intervention, drawing a graph of an imagined temperature function, the idea to use the theorem came up, and the problem was finally solved.

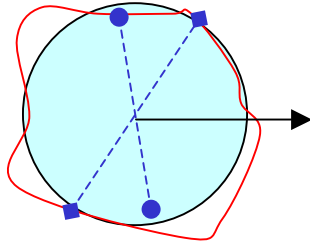
One way to intuitively visualise a possible solution to this problem is to use what might be called a *circular graph*, superimposed on the circle studied, where the temperature at each location on the circle is measured along a radial axis (adequately



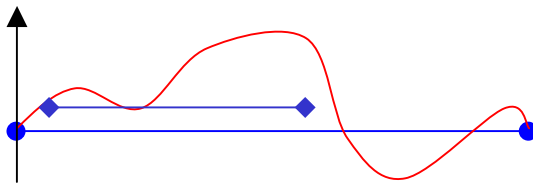
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scaled; i.e. a polar coordinate model). This graph (see figure below) shows a solution at diametrically opposite points on the circle where the corresponding points on the graph are equidistant from the center of the circle (for example at the square marked points, while the circle marked is not a solution).



The graph may be “stretched out” to the standard Cartesian form (see figure below, with a simplified graph), where a solution is found by moving a horizontal segment of half the interval (i.e. circle perimeter) with its left end point following the temperature graph until its right end point (by necessity, due to continuity) also hits the graph.



An observation to make here is that a formalisation (mathematisation) of a problem not immediately identified as a known ‘kind’ of problem, with a ready made solution at hand, is out of reach when an intuitive picture of a possible solution is missing. Now, in this case, given a graphical picture of the temperature function has been sketched, an intuitive solution can offer a cognitive support for the formal symbolic solution using the intermediate value theorem.

Rather than trying to replace intuitive notions that interfere with understanding new concepts, such as the concept of limits as introduced more formally in calculus courses, Mamona-Downs (2002) suggests, for the limit concept, a three-step teaching procedure that builds on the didactic principle of basing teaching on students’ tacit intuitive views by making them visible and conscious:

- (i) Initiating and developing intuition through raising issues in a classroom discussion environment.
 - (ii) Introducing the formal definition and to analyze it in tandem with the issues in (i). Introducing a particular representation.
 - (iii) To endorse or revoke opinions in step (i) by comparison with the formal definition, especially via the representation in (ii).
- (Mamona-Downs, 2002, p. 261)

By the key role of the graphic representation used, discussing the epsilon-delta intervals, the main cognitive tool providing the connection seems to be diagrammatic reasoning. A case study by Pinto and Tall (2002) provides data, which relates to a sequence of this kind, of a student who, rather than *extracting meaning* from the



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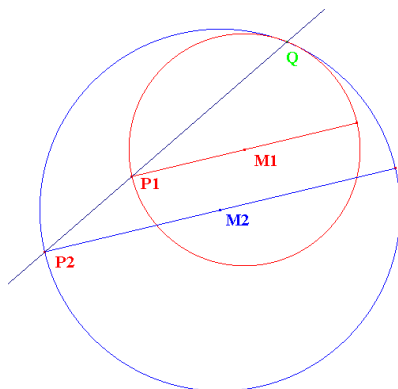
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formal definition by logical deduction of properties, “builds from his imagery, *giving meaning* to the definition by producing a highly refined image that supports his formal argument” (ibid., p. 2). The psychological impetus for the student is seen as attempts to make the concept more *concrete* to himself, by *interpreting* the definition and *exploring* the concept as a mental object, *reconstructing* his understanding. Pinto and Tall discuss this result in relation to two different forms of ‘proof’, “*thought experiment* and *propositional proof*” (ibid., p. 7), the former not requiring any formal knowledge, it is more a matter of getting convinced, often by visual support. The result of the thought experiment may then be *translated* to a propositional argument, a rigorous proof in the culture of mathematics. This move from the visual to the verbal code, the difficult part, is the “transfer from intuitive imagery to formal deduction” (ibid., p. 8). The problem session with a temperature distribution on a circle, as discussed above, provides one example of this kind of dynamics between intuitive and formal knowledge. The previous example on Newton’s method also illustrates this interplay.

Thought experiments of this kind may also be supported by artefacts such as dynamic geometry software (ibid.). As an illustration, consider the following example (see figure below).



In two circles with a tangent point (Q), two parallel diameters are drawn (M1 and M2 are center points of the circles). That the endpoints of the diameters (P1 and P2) and the tangent point (Q) are collinear² is strongly visualised by the drag mode in a dynamic geometry software, a “thought experiment” leaving no doubt about the result. From such strong “intuitive imagery” the step to a “formal deduction” within a theory may open up as an attractive challenge – how can this be? Intuition does not need the Platonic triangle. The Euclidean proof rests on reasoning from the particular diagram, and the generality is a consequence of the necessity in the argumentation (Radford, in press).

Discussion and conclusions

We have seen examples of how intuitive knowledge may provide a basis for formal knowledge to be developed. But it is not a ‘natural thing’ to mathematise, to formalise: students must be provided with explicit experience, in school settings by designing didactical situations aimed to develop formal knowledge. Real numbers, as an example, are intuitively grasped as points on the number line, even the idea of a

² This is proposition 1 in Archimedes’ Book of lemmas (Great Books, 1952, p. 561).



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‘point’ can lead to paradoxes, by intuitive tacit models of abstract points as concrete points (Fischbein, 2001). They are used in many different kinds of settings in a variety of situations (denoting magnitudes, proportions, time...) but obey the same formal rules. To ‘understand’ what real numbers are by necessity involves ‘understanding’ the formal rules for such numbers. These rules are ‘visible’ in mathematical formulas such as $a + b = b + a$ or $a \cdot (b + c) = a \cdot b + a \cdot c$. This means that for a learner, understanding of mathematical forms, as defined above, is a path that can possibly contribute to bridge the informal/formal gap. *Genetic forms* such as the above formulas for commutativity and distributivity³, have the potential of evoking the same image schemata as the ideas or activities they depict. To operate on formulas of genetic form is like operating directly on reality. They are isomorphic activities. This observation (Bergsten, 1990) may be one key to answering the classical question of why mathematics works, i.e. why results of symbolic manipulations are applicable to reality (see e.g. Kline, 1985). For this it must be noted that the more advanced form of mathematics to a large extent inherits its forms from the genetic forms of elementary arithmetic.⁴

The known difficulty of exploiting mathematical formalism in school may be due to the fact that it is often introduced as a way to communicate instead of as a tool to test or prove results. One example in relation to this was Newton’s method, where the problem itself led to a need for elaborating on the formalism. This particular problem comes from ‘inside’, i.e. from within the world of mathematics itself, though it may be modelled by an ‘outside’ situation/problem. Other problems may come from ‘outside’ the mathematical world, which for a mathematical treatment must be modelled ‘inside’. The activity may have the purpose of learning mathematics, or you solve a problem in context, by using or doing mathematics. This gives the following schema:

		Problem	
		From ‘outside’	From ‘inside’
Activity	Learning		
	Using/doing		

The role of intuitive and formal knowledge, and their interrelation, may vary within this matrix, as could be seen from some of the examples shown. Intuition seems to be a prerequisite to trigger successful modelling.

I have also touched on the role of genetic mathematical forms and diagrammatic reasoning. When the representational aspect of the formalism is taken away, opening the gap to intuitive knowledge, the formalism itself invites to diagrammatic reasoning, providing results with possible intuitive feed-back.

The discussion so far has indicated that the didactic choice is not between exploiting the gap between intuitive/informal and formal knowledge by building on the one side

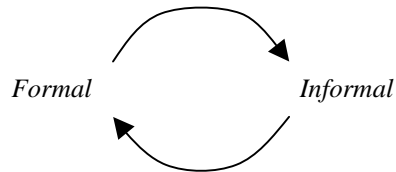
³ The structure of *genetic* forms is isomorphic to how the corresponding content (the signified) can be structured, i.e. the form is “inherited” (Bergsten, 1990). As an example, the form of $2 + 3$ is isomorphic to the form of the object arrangement oo ooo .

⁴ As an example, the common notation for composition of elements in abstract algebra, i.e. $a*b$, has the same form as the standard notation for addition and multiplication of numbers.

for the benefit of the other, but to engage in a kind of ping-pong procedure of reconstruction and refinement, i.e. the two one-way relationships,

$$\text{Informal} \rightarrow \text{Formal} \quad \text{and} \quad \text{Formal} \rightarrow \text{Informal}$$

be replaced by a circle of complementarity:



The issue whether there is a gap or not between intuitive and formal knowledge in mathematics must not be treated in isolation, taken out of context. Every mathematical activity is basically connected to some kind of problem to solve. Depending on the type of problem a range of techniques are available, for a learner in school constrained by, among other things, the curriculum, school and class organisation, and traditions of the subject. What is the purpose of using formalism or not when working with a particular problem in a particular didactic setting? This, too, relates to intuition:

“The complementarity of means and problems is what governs the evolution of our cognitions and of our intuitions in particular.” (Otte, 1994, p. 282)

At the next level, how are the discursive tools used integrated into a systematic knowledge base, and how are they justified within a coherent theory? Only in relation to such constraints and organisations can a didactic issue like the role of formalism in mathematics education be treated in a meaningful way, according to the anthropological theory of didactics (see e.g. Barbé et al., 2005). In this theoretical framework formalism is double-sided, i.e. closely related to questioning about the justification, interpretation, or description of an activity, as well as to the production of knowledge. The ‘training’ of formal knowledge out of context only adds to the difficulties students have – the formalism should have a rationale, a purpose for the student. Things do not become easier when realising that the very structure of schooling is based on putting things out of context: special houses, special age groups, put together in groups in a specially designed room with a teacher not chosen by them only teaching the knowledge and seldom practicing it. Putting students together in a classroom adds an often difficult social dimension to the teacher’s work, but also of course social benefits because of communication advantages. In this perspective, a professionally made didactic transposition which takes the intuitive-formal gap in mathematics seriously, is crucial to increase the possibilities for the teachers to be able to handle the mathematics classroom, if students developing a coherent knowledge structure is the goal. And we are not only talking about mathematics, but mathematics as a part of a general knowledge base developed by schooling.

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