Mathematics in time, the potential of calendar-mathematics in the classroom

Christoph Kirfel
College of Higher Education[1], Bergen, Norway

Introduction

For many people, especially children, the calendar we use today seems to be a nearly God-given, unshakeable and firm institution. Nevertheless our calendar system has gone through a number of changes and different cultures have developed different calendars. Even today different calendars are in use at the same time. Calendars often mirror special phenomena in nature. So the Arab calendar is a pure moon-calendar whilst the Maya calendar and the Julian or Gregorian calendars are sun-calendars. Usually calendars represent repeating events in nature such as the seasons or moon phases and try to couple these rhythms to our well known rhythm of day and night caused by the rotation of the earth. Often the problem is to synchronize two periodic movements mathematically. Understood in that manner, making a calendar is a question of mathematical modelling. As time goes by the quality of mathematical models can be questioned and often adjustments have to be made. Of courses this happened in history several times and our present Gregorian calendar is the result of a number of such adjustments.

In the lecture we get acquainted with different calendars used throughout history and we will have a look at the mathematical potential they represent. Two examples are worked out in detail. The development of the Gregorian calendar is presented and the necessary adjustments leading to our present calendar are shown. We will work out a calendar formula. Children often show some interest in calendar questions such as what day in the week they themselves or other members of their family were born on. Pupils in grade 6 or 7 can work easily with this material and be able to solve a number of calendar questions, which were rather difficult to access without that formula.

In a similar manner a moon-calendar formula is presented that may help you to find the date for celebrating Easter. Both calendar formula do not require more mathematical competence than acquaintance with multiplication tables and remainder division. At the same time they represent solutions to rather difficult modelling problems that mankind has worked on for centuries.

Calendars in general

When the human society moved from hunting and collecting communities to agrarian communities the need for calendars first arose. People wanted to know when they had to sow and when the optimal time for harvesting was. The first farmers were interested in counting days from one agricultural event to the next instead of analyzing wheather situations all the time. Thus they needed a calendar. Often the introduction of calendars was attached to religious celebrations and this is the fact today, too.

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[1] Now at the University of Bergen, Norway.

Shortly on other calendars and cultures

In Iran, Turkey and Curdistan people celebrate new year at about the end of March. This year (2004) they start the year 1382. Muhammed's escape from Mekka to Medina took place in 622 AC. At the same time they also use a religious calendar (354 days a year corresponding to exactly 12 moon cycles) and according to this calendar we have come to the year 1424 – 25. The Chinese new year takes place somewhere between 21 January and 19 February. The Hindu calendar celebrates new year in April/May. The Jewish calendar says 163 days after Easter the year before (1 Tischiri) (11 August - 10 October). Celebration of the new year has been known for 4000 years in Mesopatamia at the new moon in the middle of March. In Europe 1 January was not made the beginning of the year before in 1582.

So, we see calendars are cultural phenomena and are differently developed in different countries.

The Gregorian calendar and its history

Now, it might be important to understand our own calendar, its history and development, adjustments and reforms.

The calendar we use is based on the ancient Egyptian calendar having 365 days per year. The ancient Egyptians knew very well that this is too little and that the equinox (this is the only day in the year where the day and the night are equally long, over the whole of the earth) will move through the months and not return on a fixed date. They also knew that in a period of 4 times 365 = 1460 Egyptian years equinox will be back at its original place in the calendar. The most obvious disadvantage of the Egyptian calendar was the fact that equinox and the seasons were not coupled to fixed dates or months in the calendar. That meant that planning harvesting and sowing seasons required more than the mere calendar. Nevertheless, the Egyptians knew that and were able to correct the calendar when planning agricultural activities.

The Roman emperor Julius Caesar, advised by a Greek mathematician, took over the Egyptian calendar but introduced a leap day every fourth year. In order to adjust the calendar and equinox he introduced an exceptional year having 445 days. This so-called Julian calendar was in use until 1582 over the whole of Europe and parts of the Middle East. The first day in the year was always 1 March. March, the month of the god of war, Mars, started the year and if there was a leap day this was placed at the very end of the year. This explains why our leap days are on 29 February.

Now the average length of a year in the Julian calendar was 365.25, whilst the length of the astronomic year i.e. the time the earth uses on her orbit around the sun in order to returns to exactly the same astronomic position is 365.2422. So there was a little error. During 15 centuries this little error grew to nearly 10 days such that even the layman could see that equinox did not occur on the day the calendar predicted. So a number of scientists and clergymen - amongst them Martin Luther - proposed to Pope Gregory XIII to carry out a calendar reform. He did so in 1582 by leaving out 10 days in October 1582. The days: 5 October to 14 October did not exist, so 4 October 1582 was followed by 15 October 1582. By these means equinox was pushed back to the right place in the calendar.

In addition Gregory introduced an exception for the rule of leap years. There is no leap year in the years 1700, 1800, 1900, 2100, 2200, 2300, 2500 and so on. So, years divisible by 100 but not by 400 are excepted from the rule of leap years. Thus year 2000 in fact was a leap year and for our lifetime we need not worry about the Gregorian exception rule.

In this way the average length of the year now becomes 365, 25 - 3/400 = 365.2425 and the error made compared to the astronomic year is so small that equinox will not move more than three days in the course of the next 10,000 years. Since we leave out three leap days in a period of 400 years we had to subtract 3/400 in order to find the average length of the Gregorian year.

In the first place the calendar was reformed only in the catholic countries. The protestant countries followed later.

– Catholic countries	1582	Portugal/Italy/France/Spain/Netherlands (protestant part 1700)
– Sweden	1753	
– Norway / Denmark	1700	
– England	1752	2. September - 14. September
– Germany (cath.)	1583	Bavaria/Austria different dates (protestant 1700)
– Prussia	1612	
– Switzerland	1584/1700	
– Hungary	1587	
– Eire	1782	

Eastern Europe: Russia 1918, Greece 1923, Turkey 1914, Japan 1873, China 1912 (but not according to the birth of Christ)

One can see that the protestant and Lutheran countries introduced the new calendar much later while the Eastern orthodox countries even waited until the last century. In Norway where I work the reform was installed in 1700. For some time there were four different calendars in use in Europe.

The development of our calendar as a matter of mathematical modelling

When we build a model of a segment of the real world we usually go through a number of steps before we get to a preliminary or final result:

1. Problem formulation
2. System demarcation
3. Mathematisation
4. Mathematical manipulation
5. Evaluation and interpretation of model and model results
6. Evaluation of the validity of the model

1. Problem formulation

Here the problem formulation is not quite clear. What do we want a calendar to help us with? Find the correct starting point for seasons and help farmers in their activities, plan religious rituals and feasts, find the correct moon phases in advance, predict the day to pay interests and so on, or – the problem we shall consider - couple the rhythm of the week to that of the year and predict the day of the week for a given date?

2. System demarcation

Julius Caesar made a decision when taking over and reforming the Egyptian calendar. He decided not to couple the calendar to the moon phases and thus settled on building a mere solar calendar, meaning that the days of the equinox had to return on the same dates every year. In addition he wanted a simple predictable rule not depending on priests and clergymen, easily understood and performable over the whole of the huge Roman Empire. Thus he set the rules for the game.

3. Mathematisation

Originally the problem is a non-mathematical problem. We turn it into a mathematical problem by redefining the problem into a question of counting days in order to find the optimal time for sewing and harvesting not wanting to take weather conditions into account. Therefore the calendar can only give a kind of average result.

A calendar formula

The week, containing 7 days is a Hebrew heritage. But at the time of Caesar this rhythm was already well established in many countries around the Mediterranean Sea. How was it possible to couple the rhythm of the week to the rhythm of the years?

This question invites us to develop a formula for the day of the week for any given date. In order to do this we start numbering the days of the week. We do so by putting:

Sunday 0, Monday 1, Tuesday 2, Wednesday 3, Thursday 4, Friday 5 and Saturday 6.

So we identify 7 and 0, 8 and 1 and so on. We just look at the remainder of the number when divided by 7.

Now to calculate the day of the week we need of course the date d, we need a number m for the month (This is not going to be the one we use in usual dates), we need the year y and the century c.

We start with 1 March 2000, a Wednesday. Since March is the first month of the year (in our calculations) March is labelled 0. Since dates in April are translated by three days compared to the corresponding dates in March, April gets number 3. May is translated 2 more units, thus we get 5.

March	0	June	1	September	2	December	2
April	3	July	3	October	4	January	5
May	5	August	6	November	0	February	1

Now the last day in 1999 (following our calculation February 29, 2000) was a Tuesday (2). So we start with two, add the date and the translation of the month

$w = 2 + d + m.$

Of course only the remainder when dividing by seven is interesting. Let’s try an example: Christmas Eve, 2000. Now $m = 2$ for December and we find:

$w = 2 + 24 + 2 = 28 = 4 \cdot 7 + 0$, giving a Sunday.

We try another date: November 17 in 2000. Here $m = 0$ and we get

$w = 2 + 17 + 0 = 19 = 2 \cdot 7 + 5$, a Friday.

Now we want to include the year into the formula. Here we only look at the two last digits of the year, f. ex. 04 for the current year. For the year 2000 the formula is:

$w = 2 + d + m.$

Now a usual year contains 52 weeks and one day extra, so for 2001 the formula is

$w = 2 + 1 + d + m = 3 + d + m$

and so on. We continue in the same way and get

$w = 2+2 + d + m = 4 + d + m$ for 2002
$w = 2+3 + d + m = 5 + d + m$ for 2003
$w = 2+5 + d + m = d + m$ for 2004 (leap year)

and so on. So we just add the year y and for every fourth year we have to add an extra unit.

$w = 2 + d + m + y + [y/4].$

Here the brackets [ ] tell us, that we have to leave out what is after the decimal period. This bracket function increases by one unit every time y increases by four units. This formula already gives us a calendar valid for a whole century.

4. Mathematical manipulation

Example: let's find the date of this lecture, 6 July 2004. Here we get

$w = 2 + d + m + y + [y/4] = 2 + 6 + 3 \text{ (July)} + 4 + [4/4] = 16 = 2 \text{ (Tuesday)}$

The only thing that is specific for the 21st century in our formula is the first figure (2). For the period 1900- 1999 we have to find another starting figure. The 20th century included 100 years, 25 of which were leap years. This makes a translation of 125 days or 17 weeks and 6 days. Compared to dates in the 21st century the corresponding dates in the 20th century lie 6 days earlier in the week. So instead of figure 2 we have to use 2 - 6 = - 4. Of course we also may use + 3, since both take us to the same day in the week.

$w = 3 + d + m + y + [y/4]$ for the period 1900-1999.

5. Evaluation and interpretation of model and model results (adjustments, Gregorian reform)

The argument used above is valid for the 20th century and for all other centuries having 25 leap years. These centuries are called Julian centuries. Now, the 19th century was no Julian century since the Gregorian exception rule for the year 1900 left out one leap day. So we have to be careful. Since there were only 24 leap days we just subtract 5 more days instead of 6 and get:

$w = 5 + d + m + y + [y/4]$ for the period 1800-1899 and accordingly
$w = d + m + y + [y/4]$ for the period 1700-1799 (Denmark/Norway).

(In Great Britain, of course, we have to be extra careful with the 18th century because of the introduction of the Gregorian calendar in 1752. We have to develop two formulae, one for the first half (1700-1752) and one for the second half (1752 - 1799) of that century.

$w = 4 + d + m + y + [y/4]$ from 1700 to September 2, 1752
$w = d + m + y + [y/4]$ from 14 September 1752 to 1799.

Of course the formulae for Great Britain and Denmark coincide for the period 1752 – 1799, when the Gregorian reform was implemented in both countries.)

Going back beyond 1700, we have to take the Gregorian reform into account, which left out 10 days at the end of February in 1700 (Denmark/Norway). Thus a date between 1600 and 1699 lies -10 + 5 = - 5 = 2 days earlier in the week than the corresponding date between 1700 and 1799, so we can write:

$w = 5 + d + m + y + [y/4]$ for 1600-1699

(From 1. March 1600 to 1 March 1700 there were 100 x 52 weeks + 100 days + 24 leap days – 10 reform days = 2 days)

Going back even further in history is quite simple, since the Julian calendar was unchanged from 46BC to 1700. Since Julian centuries have 25 leap years, we have to subtract 125 days or equivalently 6 units for every century we go back in history. Equivalently we may also add one unit.

$w = 6 + d + m + y + [y/4]$ for 1500-1599,
$w = d + m + y + [y/4]$ for 1400-1499,
$w = 1 + d + m + y + [y/4]$ for 1300-1399, and so on.

Thus, for the Julian period we may write

$w = d + m + y + [y/4] - c$ where c indicates the century, f. ex. 15 for a date in 1547,

whilst we may write

$w = d + m + y + [y/4] + (0,5,3,2)$ for the Gregorian period.

Knowing that the Gregorian calendar was introduced in 1700 we use (0) from the parenthesis for dates after the reform in the 18th century, (5) from the parenthesis for the period 1800-1899, (3) for the period 1900 - 1999, as we already have seen and (2) for our century. For dates between 2100 and 2199 we start from the beginning again (0) and continue periodically with the numbers (0,5,3,2).

In Great Britain the last formula is to be used from 1752, whilst the Gregorian formula is still valid in the period before.

Thus these two formulae give us a calendar lasting from 46BC to eternity. And you may design a mnemonic rule for yourself helping you to remember the 12 translations of the months. The only thing we have to do is to learn the translation list for the months by heart and to remember that a year always starts with March 1, i.e. dates in January and February belong to the year before.

6. Evaluation of the validity of the model

Tests and comparisons with other calendars show that the formula gives us the correct day of the week for any date in a reasonable perspective of time, future or past. Further adjustments will be so small that we do not need additional leap days for several thousand years. Corrections are performed by subtracting a certain amount of seconds from the last day in the year on 31 December automatically from synchronized electrical clocks.

A moon calendar as a mathematical model

The origin of Easter is to be found in the Jewish tradition. In the Old Testament the pasha-ritual was a vernal agricultural ceremony. The feast started at the first full moon after the vernal equinox and lasted for a week. But after the exodus of the Jews from Egypt, the Jewish people no longer being prisoners of the Pharaoh, the feast gained a new element to its content. It became an anniversary of the exodus from Egypt. The New Testament couples the celebration of Easter to the death of Jesus Christ on Good Friday and his resurrection the Sunday after.

When Christians all over the world eventually had started to use the Julian calendar – a calendar that was built as a mere solar calendar – it had become difficult to calculate the date of Easter since this date was dependant on moon phases. In addition they did not want to celebrate Easter on the same date as the Jews. So there were lots of confusion and difficult claims to fulfil. In the meantime, Church leaders gathered at Nicea in 325 made it clear that

Easter Sunday is the first Sunday after the first full moon after 21 March. If this full moon is on a Sunday, Easter Sunday is to be celebrated at the following Sunday. Thus Easter Sunday will always lie between 22 March and 25 April.

In the time directly after the gathering in Nicea this rule was not followed, but after a long argument between the Alexandrian Church being responsible for the calculation of the date of Easter, and the Roman Church, Dyonysius Exigus worked out an Easter table, which after a while (8th century) was used in all Christian countries. Thus there was peace about the calculation of the date of Easter lasting for some 800 years, up to the Gregorian reform.

Now, it is very complicated to calculate the moon phases exactly since the moon sometimes moves faster than at other times. On the other hand small deviations in the calculations of the date of the full moon may have a huge effect on the date of Easter. To simplify the calculations the starting point was no longer the astronomically observable moon phases but rather a theoretical moon having a smooth, continuous movement, coinciding not too badly with the astronomic data. This theoretical moon may sometimes deviate from the moon you can see at night. It is a kind of average moon. This model is called the cyclic calculation of the moon phases.

Already the Greek, Methon in 432 BC., had observed that 235 cycles of the moon correspond very well to 19 years.

Following Ptolomaeus one cycle of the moon lasts 29.5305 days. Therefore 235 moon cycles fill up 6939,689 days, whilst 19 Julian years have a total of 6939, 75 days. The difference is so small that in the course of 1000 years the accumulated difference will be not more than 3 days. This means that every nineteenth year, e.g. the full moon will return on the same calendar dates in the year.

Now a usual year having got 365 days is 11 days longer than 12 full moon cycles (354 days). Here we approximate a moon cycle by 29.5 days ($12\cdot 29.5 = 354$). This means that the Easter full moon occurs 11 days earlier for every year. In the cyclic calculation of the date of Easter we do not in fact take leap years into account. If we cross the Easter border (21 March) we have to go to the next full moon 30 days later. If we do so 19 times after each other the date of the Easter full moon has diminished by $19\cdot 11 = 209$ days. In order to make this number a whole number of moon cycles the last decrease was set to 12 days instead of 11 days (“saltus lunae”, the jump of the moon), so that the moon phases after 19 years were back to the same calendar dates ($19\cdot 11+1 = 210 = 7\cdot 30 = 7$ moon cycles), such that the formula fit to the observations of Methon and Ptolomaeus. Since the moon phases fall on the same dates after 19 years it is enough to know the dates of the Easter full moon for 19 subsequent years.

Year	**Easter full moon**	**Year**	**Easter full moon**
1900	14 April	1910	25 March
1901	3 April.	1911	13 April = 14 March +30
1902	23 March	1912	2 April
1903	11 April = 12 March +30	1913	22 March
1904	31 March	1914	10 April = 11 March + 30
1905	18 April (19 April = 20 March +30, dating back gives 18 April)	1915	30 March
1906	8 April = (19 – 11) April	1916	17 April (18 April = 19 March +30 dating back gives 17 April)
1907	28 March	1917	7 April = (18-11) April
1908	16 April = 17 March +30	1918	27 March
1909	5 April	“Saltus lunae”	**12** days earlier (15 March +30 = 14 April)

Now, the Nicea gathering decided that the Easter full moon must not occur before 21 March. This means that the Easter full moon has to lie between 21 March and 19 April, 30 days after.

The church also decided that a full moon falling on 19 April has to be dated back to 18. April. In the same way an Easter full moon falling on 18 April has to be dated back to 17 April in order to avoid that two Easter full moons fall on the same date in the course of 19 years. This fact is valid for the period 1900 – 2199. In 1900 The Easter full moon was on 14 April. The following year the Easter full moon occurred 11 days earlier (14-11 = 3 April). Every year the date of the Easter full moon decreases by 11 days, with the exception of the 19th year, where the decrease is 12 days. Then we are back at the original date 14 April.

Now dates for the moon phases repeat themselves after 19 years and we can calculate all dates of Easter full moons starting with the table above.

In this presentation we give a formula covering 300 years containing the crucial dates for our lifetime, the period from 1900 – 2199. We put y equal to the number of years after 1900, i.e. $y = 84$ for 1984 and $y = 104$ for 2004. Then we easily find how much the original Easter date (14 April in 1900) has to be decreased in order to find the new date for the full moon.

$D = 14\text{ April} - 11\cdot y - [y/19]$

Here the term $-11\cdot y$ gives a decrease of 11 days for every year and the term $-[y/19]$ gives an extra decrease of one unit for every nineteenth year (“saltus lunae”).

The number $-11\cdot y - [y/19]$ is of course negative and we have to add a number of whole moon cycles (30 days) in order to make the date fit into the period from 21 March to 19 April.

Example: 2004. Here $y = 104$ and we calculate

$-11\cdot 104 - [104/19] = -1144 - 5 = -1149.$

Now $-1149 + 38\cdot 30 = -9$, so the Easter full moon in 2004 occurred on $D = 14 - 9$ April = 5 April.

To give a formula that is correct for a larger period of time one has to take care of a number of additional corrections and adjustments, which are too complicated to present in this short lecture.

We see that a great number of the criteria for a modelling process are present in working with the moon calendar. The model gives only an answer to a part of the original problem. In order to attack the problem the question had to be redefined and we had to leave the real moon and look at an imaginary, theoretical moon moving smoothly across the sky. Equality notions are used in a strange manner when working with dates and congruences.

Conclusion

The examples in the lecture show how calendar mathematics and working with calendar formulae may illustrate mathematical modelling. Starting with a non-mathematical question we have to redefine the problem in order to mathematize the problem and make it accessible for an algebraic attack. Here we already lose part of the original question. A number of approximations are made on the way. Leap days and "saltus lunae" are examples of how to make different cycles meet in the end. In fact different approximations are chosen along the way. Some of them fit to long-term considerations others to short-term considerations. The result often is a preliminary one and may be subject to reforms and changes as we have seen.

References

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