On the relationships between informal out-of-school mathematics and formal in-school mathematics in the development of abstract mathematical knowledge

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Introduction

The relationship between mathematics[1] and reality has always been both intricate and intriguing, complicated and interesting. Perhaps we will never be able to analyze it completely.

Jokingly, we might call it 'a love/hate relationship' since mathematics, though nourished by the real world, soon separated from it, due to its special nature, only to return to real experience in due time to pick up new problems and examples or to find new applications.

As to didactics, the fact that this relationship is sometimes denied and at other times stressed, with no explanation of the reasons for these choices, makes it difficult for students to know whether or not it is permissible for them to exploit their everyday knowledge in approaching mathematical problems.

Furthermore in the teaching of mathematics there is a strong tendency to emulate the practice of academic mathematicians in showing only the finished product of mathematical research rather than highlighting the process of creation. In other words, the results are presented as if the audience consisted of expert colleagues expecting a refined and elegant presentation purged of all the 'dirty work' which was needed to produce the results. It is precisely the latter which could be illuminating and interesting for students (Bonotto, 2001a).

In and out of school mathematics

In many current reform documents relating to mathematics education, a strong plea is made for making problem solving in school mathematics more closely related to the experiential worlds of children by using more complex and more authentic problem situations in the mathematics lessons.

The connection between in- and out-of-school mathematics is not easy to make because the two contexts differ significantly. Just as mathematical practice differs in and out of school (Lave, 1988; Nunes, 1993) so does mathematics learning (Resnick, 1987). Masingila, Davidenko, and Prus-Wisniowska (1996) outlined three key differences between in- and out-of-school practices (goals of the activity, conceptual understanding, and flexibility in dealing with constraints). In out-of-school mathematical practice in particular, people may generalize procedures within one context, but may not be able to generalize to another, since problems tend to be context specific. Generalization, an important goal in school mathematics, is not usually a goal in out-of-school mathematics.

In common teaching practice the habit of connecting mathematics classroom activities with everyday-life experience is still substantially delegated to word problems.

[1] Intended here as academic mathematics, i.e. the mathematics practiced by a particular cultural group, namely professional mathematicians

But beside representing the interplay between these two contexts, word problems are often the only means of providing students with a basic sense experience in mathematization and mathematical modeling.

Recent studies have documented that the practice of word problem solving in school mathematics promotes in students

- the exclusion of realistic considerations
- a “suspension of sense-making” (Schoenfeld, 1991)

and rarely reaches the idea of mathematical modeling and mathematization (see Verschaffel et al., 2000, for a review of these studies).

Primary and secondary-school students, and also student-teachers, tend to ignore relevant and plausible familiar aspects of reality and exclude real-world knowledge from their observation and reasoning (Verschaffel, De Corte & Borghart, 1997).

Several studies point to two causes for the abstention from using everyday-life knowledge:

- textual factors relating to the stereotyped nature of the most frequently used textbook problems
  “When problem solving is routinised in stereotypical patterns, it will in many cases be easier for the student to solve the problem than to understand the solution and why it fits the problem” (Wyndhamn and Säljö, 1997).

- presentational or contextual factors associated with practices, environments and expectations related to the classroom culture of mathematical problem solving
  “In general the classroom climate is one that endorses separation between school mathematics and every-day life reality” (Gravemeijer, 1997).

Furthermore, it is noted that the use of stereotyped problems and the accompanying classroom climate relate to teachers’ beliefs about the goals of mathematics education (Verschaffel, De Corte & Borghart, 1997).

Finally, in our opinion, the practice of word problem solving is relegated to classroom activities, having meaning and location, in terms of time and space, only within the school. Rarely will students encounter these activities in this form outside of school.

This indicates a difference in views on the function of word problems in mathematics education. The researchers, and probably the drafters of new curricula, such as the Italian one, relate word problems to problem solving activity and applications. For student-teachers (and probably teachers in general) word problems are nothing other than exercises in the four basic operations, which also have a justification and suitable place within the teaching of mathematics, though certainly not that of fostering a process of providing students with a basic sense experience in “*mathematization*” or “*realistic mathematical modeling*” in the sense of for example “*both real-world based and quantitatively constrained sense-making*”, Reusser & Stebler, 1997.

> “…there are abbreviated and restricted links between mathematics and reality which are much more frequently found: On the one hand a direct application of already developed “standard” mathematical models to real situations with a mathematical content, on the other hand a “dressing up” of purely mathematical problems in the words of an other discipline or of

> everyday life. Such word problems often give a distorted picture of reality." (Blum and Niss, 1991).

Also Freudenthal's position on the matter is very clear; he stresses the pseudo-isomorphism implicit in these problems and worries that this practice may prompt an anti-mathematical attitude:

> "The context [of the butcher problem, author's note] is the textbook, rather than reality proper, or in other words, it portrays a world of pseudo-isomorphisms. In the textbook context each problem has one and only one solution: there is no access for reality, with its unsolvable and multiply solvable problems. The pupil is supposed to discover the pseudo-isomorphisms envisages by the textbook author and to solve problems, which look as though they were tied to reality, by means of these pseudo-isomorphisms. Wouldn't it be worthwhile investigating whether and how this didactic breeds an anti-mathematical attitude and why the children's immunity against this deformation is so varied?" (Freudenthal, 1991).

We deem that if we wish

a) real problems arising from the children's real experiences, so that students may connect reasoning, practices and experiences both in- and out-of-school in a back and forth process,
b) situations of realistic mathematical modeling, in problem solving activities,
c) that the students learn mathematics by mathematizing, by overcoming the dichotomy between mathematics as an activity and mathematics as a body of knowledge,

we have to make changes.

1. We have to replace the type of activity aimed at creating interplay between in- and out-of-school mathematics with more realistic and less stereotyped problem situations (based on the use of concrete materials or suitable materials).

2. We must endeavour to change students' beliefs and attitudes towards mathematics; this means changing teachers' conceptions, beliefs and attitudes as well.

3. And finally, a directed effort has to be made to change the classroom culture.

In this lecture, based on results obtained in several teaching experiments, we discuss how these changes can be brought about at primary school level through classroom activities which are more easily related to the experiential world of the student and consistent with a sense-making disposition.

The activities make extensive use of cultural artefacts that could prove to be useful instruments in creating a new tension between school mathematics and the real world with its incorporated mathematics.

In particular we will show, through a study, that is a paradigmatic example, how suitable cultural artefacts and interactive teaching methods can play a fundamental role in bringing students' everyday-life experiences and informal reasoning into play.

In our approach in- and out-of-school mathematics, even with their specific differences, in terms both of practices and learning processes, are not seen as two disjoint and independent entities. Furthermore we think that the conditions that often make out-of-school learning more effective can and must be re-created, at least partially, within classroom activities. Indeed, though there may be some inherent differences between the two contexts, these can be reduced by creating classroom situations that promote learning processes closer to those arising from out-of-school mathematics practices.

Our approach

The critical problem of how to manage the relationship between informal out-of-school and formal in-school mathematics has been the subject of our studies for some years.

In agreement with the Realistic Mathematics Education perspective of the Dutch school of thought (Treffers, 1993; Gravemeijer 1994), we believe that formal mathematics is not something "out there" with which the student has to connect. Instead, formal mathematics is seen as something that grows out of the students' activity, and comes to the fore as a natural extension of the student's experiential reality.

The students are expected to develop formal mathematics by way of mathematizing their own informal mathematical activities; they have to experience formal mathematics no differently from informal mathematics:

> "we can distinguish between formal and informal mathematics by denoting formal mathematics reasoning as a form of reasoning that builds on arguments that are that are located in the newly formed mathematical reality. Seen this way, the distinction between informal and formal mathematics is a relative distinction – a distinction that is relative to a certain topic and that can be made especially from an observer perspective" (Gravemeijer, 1999).

The progressive mathematization should lead to algorithms, concepts and notations that are rooted in a learning history which starts with students' informal experientially real knowledge. The idea is not only to motivate students with everyday-life contexts but also to look for contexts that are experientially real for the students and can be used as starting points for progressive mathematization (Gravemeijer, 1999).

Furthermore we stress that

the process of bringing the real world into mathematics

by starting from student's everyday-life experience, is fundamental in school practice for the development of new mathematical knowledge, but it turns out to be a condition which is necessary but not sufficient in itself to foster

> "a positive attitude towards mathematics, intended both as an effective device for discovering and critically interpreting reality, and as a fascinating thinking activity"

as is stressed in the Italian primary school program.

We contend that these educational objectives can be completely fulfilled only if we also foster in school practices the process of bringing mathematics into reality.

In other words, besides
mathematizing everyday experience
it is necessary
'to everyday' mathematics.

We believe that this can be implemented in the classroom by encouraging students to analyze '*mathematical facts'* embedded in appropriate '*cultural artefacts*'.[2]

In other words, we want to encourage the children to recognize a wide variety of situations as mathematical situations, or more precisely as "mathematisable" situations, since a great deal of mathematics is embedded in everyday life.

In this way we can multiply the occasions when students encounter mathematics outside of the school context.

The cultural artefacts we introduced into classroom activities, e.g.

- supermarket bills to introduce some aspects of the multiplicative structure of decimal numbers, [Bonotto, 2001a; Bonotto (2005)],
- a ruler to foster children's decimal number understanding [Bonotto, 2001b],
- a cover of a ring binder to introduce the concept of surface area [Bonotto, 2003a],
- a weekly TV guide to develop the concept of equivalence between time intervals expressed in different ways [Bonotto, 2003b],
- an informational booklet issued by "Poste Italiane" to estimate and discover area and length dimensions of some envelopes [Bonotto & Ceroni, 2003],

are concrete materials which children typically meet in real-life situations.

We have therefore offered the opportunity of making connections between the mathematics incorporated in real-life situations and school mathematics, which although closely related, are governed by different laws and principles.

These artefacts are relevant to children; they are meaningful because they are part of their real life experience, offering significant references to concrete, or more concrete, situations.

This enables children to keep their reasoning processes meaningful and to monitor their inferences; so they can off-load their cognitive space and free cognitive resources to develop more knowledge (Arcavi, 1994).

Roughly speaking

> "in the ticket, which is poor in words but rich in implicit meanings, the situation is overturned with respect to the usual buying and selling problem,

[2] Cultural artefacts beside embodying the intellectual history of a culture also incorporate many theories the users accept, albeit unconsciously. "Artefact and conventions are cultural forms that have been created over the course of social history which also figure into the goals that emerge in cultural practices" (Saxe et al. 1996). Their use mediates the intellectual activities, and - at the same time - enables and constrains human thinking. Through these subtle processes social history is brought into any individual act of cognition (Cole 1985). Thus learning mathematics is not excluded.

which is often rich in words but poor in meaningful references” (Bonotto, 2001).

The dual nature of these artefacts, that of belonging to the world of everyday life and to the world of symbols, to use Freudenthal’s apt expression, makes possible the movement from the situations in which it is usually used to the underlying mathematical structure, as well as the reverse process, from the mathematical concepts to the real world situations; this is in agreement with ‘*horizontal mathematization*’, Treffers 1987. An essential property of artefacts, which supports their bilateral influence and offers common bases to culture and discourse, is their being ideal (conceptual) and material.

But a different use of these same artefacts, with certain modifications - for instance removing some data present in the artefacts, as for example in Bonotto (2005) - supported the opportunity to favor also ‘*vertical mathematization*’, from concepts to concepts, see Figure 1, although only in a weak sense, given the grade level of the students.

‘*horizontal mathematization*’

real world situations ⇔ mathematical concepts

⇕ “*vertical mathematization*”

mathematical concepts

Figure 1

This occurred when symbols, i.e. embedded mathematical facts, became objects to be put in relationship, modified, manipulated, and reflected upon by the children through property noticing, conjecturing, and problem solving.

In this way the cultural artefact can be used to introduce new mathematical knowledge through those special learning processes that Freudenthal, 1991 called ‘*prospective learning*’ or ‘*anticipatory learning*’.

In this new role these artefacts also may become real “*mathematizing tools*”, capable on the one hand of creating new mathematical goals, and on the other of providing pupils and students with a basic sense experience in mathematization which preserves the focus on meaning found in everyday situations.

The use of these artefacts allows the teacher to propose many questions, remarks, and culturally and scientifically interesting inquiries. The activities and connections that can be made depend, of course, on the students’ school level.

These artefacts may contain different codes, percentages, numerical expressions, and different quantities with their related units of measure, and hence are connected with other mathematical concepts and also other disciplines (chemistry, biology, geography, astronomy, etc.).

It could be said that the artefacts are related to mathematics (and other disciplines) as long as one is able to form these relationships.

Furthermore we ask children

- to select other cultural artefacts from their everyday life,
- to point out the embedded mathematical facts,
- to look for analogies and differences (e.g. different number representations),
- to generate problems (e.g. discover relationships between quantities).

So we can present mathematics as a means by which to understand the real world. We deem that in this way we can enable students to become involved with mathematics, to break down their conceptions of a remote body of knowledge and to develop a positive attitude towards school mathematics.

The basic characteristics of the teaching/learning environment

Beside the use of suitable cultural artefacts discussed above, the teaching/learning environment designed and implemented in our classroom activities is characterized by:

the application of a variety of complementary, integrated and interactive instructional techniques;
an attempt to establish a new classroom culture also through new socio-mathematical norms.

Regarding the first point, most of the lessons follow an instructional model consisting in the following sequence of classroom activities:

a) a short introduction to the class as a whole;
b) an individual written assignment where students explain the reasoning followed, the method used, or the strategy applied;
c) a whole-class discussion (the results obtained through personal reflection and elaboration were discussed collectively, sometimes corrected, and then systematized and re-elaborated);
d) the creation of a collective written text (comprising the clearer and more convincing explanations emerging from the whole-class discussion) aimed at socialization of the knowledge acquired.

We consider that the interactivity of these instructional techniques (individual written report, group discussion, collective text) is essential because of the opportunities to induce reflection as well as cognitive and metacognitive changes in students. This process may be very important for teachers also, since it enables them to recognize and analyze individual reasoning processes that are not always explicit. In a successive phase, comparing the different answers and strategies, children's first attempts at generalizing, and further remarks made during the discussion, lead to collectively drawing up a text aimed at socialization of the knowledge acquired. This text, which completes the activity, is necessary to systematize the mathematical structures underlying the classroom activity; it is the phase of institutionalization of the mathematical concepts and processes shared by the whole class.

As regards the point 2, as in the RME perspective, in our experience too, students are expected to approach an unfamiliar problem as a situation to be mathematized, not primarily as a situation for application of ready-made solution procedures. This does not mean that the student's knowledge of solution procedures does not play a role, but

ICME 10 2004

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Regular Lecture

the primary objective of the student would be to make sense of the problem. In practice, it will often be a matter of shuttling back and forth between the interpretation of the problem and a review of possible suitable procedures or results.

At the same time, the teacher is expected to encourage students to use their own methods, exploring their usefulness and soundness with regard to the problem. The teacher should stimulate students to articulate and reflect on their personal beliefs, misconceptions and problem-solving strategies. Other possible strategies for solving the same problem when it appears next are emphasized and students are encouraged to make comparisons between strategies.

Furthermore according to the socio-constructivist perspective, these socio-mathematical norms are not predetermined criteria introduced into the classroom from the outside. On the contrary, these normative understandings are constructed and continually modified by the pupils and the teacher through their ongoing activities and interactions. The development of people's mathematical reasoning and sense-making processes is seen as

> "inseparably interwoven with their participation in the interactive constitution of taken-as-shared mathematical meanings and norms" (Yackel & Cobb, 1996).

The study

In this quasi-experimental study, see Bonotto 2003b, we decided to exploit as artefact a TV guide from a well-known weekly magazine in order to:

extend students' capacity to calculate from base 10 to base 12, 24 or 60,
develop the concept of equivalence between time intervals expressed in different ways (days, hours, minutes),
introduce informally the concept of fractions.[3]

The children in the classes involved did not know how to carry out calculations with hours and minutes, however they all knew how to add and subtract in base 10, and remembered from the previous school year that an hour is made up of 60 minutes.

To check students' familiarity with TV program guides, the experience was preceded by a phase in which children were asked to bring to class magazines and daily papers they usually use to choose TV programs. Magazines read and used only by parents, that led however to discussion within the family, were also accepted. It was found that the timetable of television programs, directly or indirectly, is part of the experiential reality of the children involved in the experience. All said that they knew the starting time and duration of their preferred programs, and that they were able to regulate TV viewing with their daily activities.

Participants

The study was carried out in two third-grade classes (children 8-9 years of age) in a suburb of the city of Padova (Padua) by the official logic-mathematics teacher, in the

[3] "Ratio, for instance, has profound visual roots, which can be arithmetised early on by estimate and measurement. There are many informal opportunities in contexts for common sense ratio in everyday language before it is dealt with more systematically and formally. Long before fractioning the traditional "cake", the clock dial is divided according to halves and quarters of an hour, which, unlike the cake segments, have an existence of their own" (Freudenthal, 1991).

presence of a researcher-teacher. The first class consisted of 20 pupils (10 girls and 10 boys), the second class of 21 (10 girls and 11 boys). In each class there were three children with learning difficulties, and two in the first class and one in the second who displayed demotivated behaviour towards school activities.

As a control, two third-grade classes [4] (children 8-9 years old) were chosen from another area of Padova, in keeping with the following criteria: i) the congruence of socio-cultural background, ii) the homogeneous level of performance with the two classes involved in the teaching experiment (as confirmed by the outcome of the pre-test) and finally iii) the use by teachers of a traditional teaching method.

Procedure

After time to collect, read and comment on the various TV guides gathered by the children, it was decided that all children should work on the same TV guide in order to be able to manage and organize the classes better. The guide included in a weekly supplement of a well-known daily paper was chosen rather than a specialized magazine because of the simpler, compact and ordered structure of the television programs of any one day. This guide also has a section, on the two following pages, dedicated to a review of the films to be televised, where the starting time, duration, but not the finishing time, can be found. Among the details presented is the date of production from which it is possible to calculate the age of the film, see Figure 2.

Then it was decided to subdivide the teaching experiment into 10 sessions, at weekly intervals, 8 sessions of one hour each and 2 of two hours each, for a total of 12 hours.

The first 5 sessions were dedicated to familiarization with the artefact, classification of the various programs according to typology (news, cartoons, films, etc) and to discovering the mathematical facts included, selecting from the many that were found.

The remaining 5 sessions concerned 2 experiences, the first of 3 hours and the other of 4, involving two different opportunities offered by the artefact chosen.

In the first experience, using the table of television programs, the children were asked to organize their day, and then the week, keeping in mind their activities and commitments, and not exceeding an hour and a half of television a day.

The second experience, which took place in 2 two hour sessions, was aimed at reading and interpreting the numerical data in the artefact used - this time the reviews of the two films. The aim also included calculating the duration of the two films in minutes and converting them to hours, and finally establishing a strategy to find the finishing time of the film (see Figure 2 for the requirements of the second experience). The children were then left free to discover other spontaneous scientific dilemmas, for example the age of the film.

Each session of these two experiences was divided into three phases. In the first, each pupil was given an assignment to carry out individually. The children were asked to answer all the questions in writing, individually. In the second phase, the results obtained through personal reflection and elaboration were discussed collectively, sometimes corrected, and then systematized and re-elaborated. The third was aimed at

[4] The first control class consisted of 20 pupils (12 girls and 8 boys), the second control class of 19 (11 girls and 8 boys).

the elaboration of a collective written text comprising the clearer and more convincing explanations emerging from the whole-class discussion.

Questions asked: *Make an evaluation of the information presented in the film review, in particular the time the film ends. Write down the procedure you used.*	Film *Courage* *** Rete 4 / time: 16.00 Producer... With... Review… Comedy Italy 1956 Duration 95’ Ó
Questions asked: *Make an evaluation of the information presented in the film review, in particular the time the film ends. Write down the procedure you used.*	Film *The Secret of the Old Forest* ** Channel 5 / time: 0.30 Producer... With... Review… Fairytale Italy 1993 Duration 134’ Ó

Figure 2

As far as the control classes were concerned, the class teachers dedicated, within the same time period, exactly 12 hours, to class activities regarding reading and calculation of time duration measured in hours and minutes, according to the modality and techniques normally used in elementary school.

Data

The research method was both qualitative and quantitative.

The qualitative data consisted of students’ written work, audio recordings and field notes of classroom observations and audio recordings of mini-interviews with students.

The quantitative data was collected by means of pre- and post-tests, administered to the two experimental classes as well as the other two control classes. The two tests were constructed by the official class teachers, not the researcher-teacher, by taking some items normally used in the bimonthly tests utilized by the same teachers.

Both the pre- and post-tests were organized in such a way as to evaluate the effects of learning on time duration (part 1) and fractions (part 2). The structure of items remained basically the same in the pre- and post-test, although post-test items included more difficult data or figures.

Research questions and hypotheses

In terms of learning processes, it was decided to continue gathering information on the way a particular artefact i) could play a fundamental role in bringing students’ out-of-school reasoning experiences into play, by creating a new tension between school mathematics and everyday-life knowledge with its incorporated mathematics, ii) could be utilized as a motivating stepping-stone to launch, at a first stage, new mathematical concepts or algorithms.

The first general hypothesis was that the children in the teaching experiment class[5] were able to grasp the calculation of hours and minutes and the equivalence between time intervals expressed in different forms (days, hours, minutes) more effectively, compared with the control class, who received a more traditional teaching method (hypothesis I).

It was also hypothesized that using the clock face, which is divided into half and quarter hours, would allow participants to work out the concepts related to fractions according to "*prospective learning*" (hypothesis II).

Furthermore, we hypothesized that, contrary to the practice of word-problem solving documented in the literature, children in this teaching experiment would not ignore the relevant, plausible and familiar aspects of reality, nor would they exclude real-world knowledge from their observations and reasoning (hypothesis III).

Finally children would also exhibit flexibility in their reasoning, by exploring different strategies, often sensitive to the context and quantities involved, in a way that was meaningful and consistent with a sense-making disposition and closer to the procedures emerging from out-of-school mathematics practice; children would also activate problem posing procedures (hypothesis IV).

Some results

Some early results from the second experience are reported.

From the first film review all the children except one, were able to elaborate in their own words the information regarding starting time, channel, year of production, etc (see Fig. 2).

Of 41 children, 28 noted and commented on the judgment of the review (for example "*as it has 3 stars, it means it is good*"). Some noted that the film had a green symbol (children's viewing) and 16 children calculated the age of the film, even if they were not explicitly asked to do so, therefore activating a problem-posing procedure. 26 children worked out the equivalence "*95 minutes = 1 hour and 35 minutes*" and 29 mentioned the time the film finished.

We note the case of a child, whom we will call Emanuele, a repeating student with serious schooling demotivation and learning difficulties. At the end of the first phase, the written report, he handed in a blank sheet. It was therefore decided to test his knowledge and thought processes by an individual interview. We discovered that he knew how to read the data in the artefact and how to correctly work out the equivalence by referring to his preferred interest, football. In fact he knew that the duration of a football match is 90 minutes, and that it corresponds to an hour and a half because he always watches sports programs with his father, the most well-known of which is called "*novantesimo minuto*" ("*ninetieth minute*"). Therefore, 95 minutes for him was equivalent to an hour and half plus 5 minutes. The case of Emanuele therefore confirmed our third hypothesis.

Regarding the task for the second review (see Fig. 2), we found that 15 children calculated the age of the film, 20 correctly interpreted the time 0.30 as "*half past midnight*", 27 children worked out the equivalence between minutes and hours correctly, and 21 worked out the time the film would have finished. During the whole-class discussion the entire class participated with great interest in the problem raised that 0.30 and 24.30 may not be the same times.

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[5] Thanks to the opportunity they had to refer to a concrete reality (the cultural artefact), to explore their strategies and to compare them with those of their schoolmates

Some significant extracts are presented from written work regarding the finishing time of the film. These show the activation of strategies sensitive to the context and quantities involved and also the emergence of problem posing activities.

> Claudia*: I pretended that the film started at exactly 0. I put the 30 minutes to one side. I added 2 hours and that makes 2. I added the 30 minutes and so I got 2 and 30. I added the 14 minutes and so arrived at 2 hours and 44 minutes.*

Claudia tried to simplify the data as much as possible to be able to calculate with greater certainty. The explanation was extremely clear, expressed in the language and terminology normally used by children, and for these reasons during the class discussion it led to curiosity, attention, understanding and participation by classmates who were unable to find the finishing time.

Gregorio's protocol included the following:

> *1) I found 2 hours and 14 minutes in this way: 60+60=120+14=134. 2) To arrive at 2.44 it was 0.30+2 hours=2.30+14=2.44. 3) To get 8 years we worked out 1993 to arrive at 2001 makes 8 years. We can see 51 minutes of the film "The executors".*

It can be seen that in the end Gregorio faced a spontaneous dilemma with a film whose review was next to the one assigned and whose viewing time partially overlapped. He posed the question

> *Once the film "The Old Forest" is finished, how much of the film "The Executors" can I watch?*

This shows how the use of an artefact may evoke situations that are in fact experienced, activating the ability to pose and resolve problems.

On the basis of the qualitative results we can say that this experience has reinforced knowledge of the hours in the day and led to calculation in base 60 by means of an informal, non-conventional, procedure on the basis of intuition linked to the context or the quantities involved. Among the children's protocols, attempts at formalizing calculation in rows and columns also appeared.

As far as the outcomes of the pre-tests and post-tests are concerned, the errors in the experimental group diminished by 46% overall, while those of the control group remained more or less stationary (hypothesis I). The two parts of each test are outlined, that is the first part testing reading ability, calculation of hours and the equivalence between time intervals expressed in different forms, and the second part regarding knowledge of the concept of fractions. It emerged that the greater improvement in the experimental group's performance is relative to the abilities tested in the second part of the test, where the concept of fractions was evaluated. There was in fact a 63% reduction in errors in the case of the experimental group, while errors increased for the control group (hypothesis II).

From the results it appears that the teaching experiment had a significant positive effect on achieving learning goals, in particular enhancing and understanding the calculation of hours and minutes and the equivalences between time intervals expressed in different forms, and even more enhancing a first approach to the concept of fractions in a way that is meaningful and consistent with a sense-making disposition.

This was not the case in the control group where an increase in errors was found in the second part of the test. It could be supposed that the control group, who received a more traditional type of teaching, may have acquired general algorithmic procedures and formal rules, but these were not well mastered and therefore did not improve performance.

The first two research hypotheses were therefore confirmed.

It was also confirmed by the qualitative results that using the TV guide did not activate rigid and general algorithmic procedures but rather specific heuristics, that have an inner consistency and value. The strategies were flexible, local and sensitive to number sizes (hypothesis IV), and were such that children often made reference to parts of the hour (half and quarter hours) to be able to manage calculations better, and in a way that was meaningful and consistent with a sense-making disposition. This aspect made them more sensitive to the concept of fractions according to *prospective learning* and therefore led to the distinct improvement (63%) by the experimental classes in the second part of the post-test.

We can say that in our teaching experiments, contrary to the practice of word-problem solving in school mathematics, children did not ignore the relevant and plausible, familiar aspects of reality, nor did they exclude real-world knowledge from their observation and reasoning (hypothesis III).

Conclusion and open problems

In our view, the positive results obtained in this study, as in our other studies, can be attributed to a combination of closely linked factors:

a) the use of suitable cultural artefacts that represent a connection with out-of-school reality or are tied to real-world situations, that allow children good control of inferences and results, and make a connection between symbols and their referents;
b) the introduction of particular socio-mathematical norms that played an important role in giving meaning to new mathematical knowledge (prospective learning) or reinforcing previous knowledge (retrospective learning);
c) systematic attention being paid to the nature of the problems and the classroom culture.

We do not suggest that the classroom activities described here are a prototype for all classroom activities related to mathematics, although in agreement with Verschaffel et al. (1999), we think that

> "the development of mathematical problem-solving, skills, beliefs, and attitudes should not emanate from a specific part of the curriculum but should permeate the entire curriculum",

as a 'problem posing' program which integrates the promising outcomes of recent studies (English, 1998).

We do believe, however, that by enacting some of these experiences, children are offered an opportunity to change their beliefs about, and attitudes towards school mathematics. Immersing students in situations more relatable to their direct experience and more consistent with sense-making, provides a means to deepen and broaden their understanding of the scope and usefulness of mathematics as well as learning ways of thinking mathematically that are supported by mathematizing situations. This allows students to become involved in mathematics and to break down their conceptions of a remote body of knowledge.

By using appropriate cultural artefacts, which students can understand, analyze and interpret, we can present mathematics as a means of interpreting and understanding reality. Teaching students to interpret critically the reality they live in, to understand its codes and messages so as not to be excluded or misled should be an important goal for compulsory education. The computer, as well as other more recent multimedia instruments, has a remarkable social and cultural impact and huge educational potential that perhaps has not yet been fully explored (Bonotto, 1999).

Obviously, the usefulness and pervasive character of mathematics are merely two of its many facets and can not by themselves capture its very special character, relevance, and cultural value; nonetheless we deem that these two elements can be usefully exploited from the teaching point of view because they can change the common behavior and attitude held both by teachers and pupils.

For a real possibility to implement this kind of classroom activities, there also needs to be a radical change on the part of teachers. They have to try:

- to modify their attitude to mathematics, which is influenced by the way they have learned it;
- to revise their beliefs about the role of everyday knowledge in mathematical problem solving;
- to see mathematics incorporated into the real world as a starting point for mathematical activities in the classroom, thus revising their current classroom practice, and
- to investigate the mathematical ideas and practices of the cultural, ethnic, linguistic communities of their pupils in order to offer them significant references to familiar situations.[6]

Only in this way can a different classroom culture be attained.

> "The main problem [regarding rich contexts, author's note] is that of implementation, which requires a fundamental change in teaching attitudes before it can be solved" (Freudenthal, 1991).[7]

Finally, the teacher has to be ready *to create* and *manage open situations,* that are continuously being *transformed* and of which he/she cannot foresee the final evolution or result. These situations are sensitive to the social interactions that are established, to the students attitudes, reactions, their ability to ask questions, to find links between in-school and out-of-school knowledge; hence the teacher has to be able to modify along the way the contents and objectives of the lesson. The teacher has to feel confident and qualified both with regard to the mathematical contents and the educational objectives

[6] In agreement with D'Ambrosio (1985), we deem that the mathematics curriculum in school should incorporate elements belonging to the sociocultural environment of the pupils and teachers, in such a way that they facilitate the acquisition of knowledge, understanding, and compatibilization of known and current popular practices.

[7] Cultural artefacts used, and the way they have been used, can be considered as "*contexts*" or "*rich materials*" in the meaning given by Freudenthal, who underlines their qualities through a comparison with structured material.

that are potentially contained in these artefacts. Thus, not all aspects of the lesson can be prepared in advance, nor can it be prepared 'from above'; rather it should be planned as various '*branches*' to be then drawn together through a process whose handling requires a fair degree of skill and effort.

In agreement with Blum and Niss (1991) and Verschaffel et al. (1999), we deem that the effective establishment of a learning environment, like the one described here, makes very great demands on the teacher, and therefore requires revision and change in teacher training, both initially and through in-service programs**.**

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