Doing ≠ Construing and Doing + Discussing ≠ Learning: The importance of the structure of attention

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RL

Regular Lecture

Abstract

My aim is to put forward the conjecture that one of the most significant attributes of human beings, namely their attention, plays a central role in teaching and learning. What matters is not only what people are attending to, but how they are attending. By offering a range of tasks for the reader I hope to provide a taste of what I mean by the structure of attention. In the period leading through Mason and Davis (1989) and Mason (1989) to Mason (1998) I distinguished between how focused or diffuse, whether single or split, and how localised attention was generally, at any moment. Here I distinguish between different types of attention specifically related to mathematics: gazing at wholes, recognising relationships, perceiving as properties and reasoning on the basis of specified properties alone. I propose that many of the mismatches between learners and teachers are due to differences in the structures of their attention, that is, not only what they are attending to, but how they are attending. This analysis is able to go beyond the observation of 'difference of perspective' or 'seeing differently'. I also propose that mathematical reasoning is difficult to teach because of the subtle but necessary shift from perceiving as properties to reasoning on the basis of specified properties alone. Of course there are strong similarities with the work of Dina van Hiele-Geldof and her husband Pierre van Hiele (1986), but there are differences. Far from finding static states and levels, interrogation of experience indicates that the shifts between ways of attending can be very rapid. The reader is invited to notice this when engaging in the tasks presented, selected from those used in the lecture.

Contextual remarks

I have observed that efforts to increase learner activity and engagement in classrooms have not made any significant difference in the large to the learning of mathematics. I am also confident that the current obsession in mathematics education with group work and discussion is not, and cannot be, a panacea. My aim in this paper is to suggest that in order to appreciate learners' experience of mathematics it is vital to become aware of how my own attention and that of learners is differently and variously structured at different times when focusing on mathematical ideas, problems and tasks.

The paper takes the musical form of prelude, theme and variations, including a reprise. The few examples for which there is space here are necessarily text-based, but what I am offering applies to any mode or mixture of modes of presentation: symbols words, diagrams, animations, interactive animations and even thought experiments. The extended paper has more examples[i] and refers to a wider range of the literature on attention.

Phenomena being addressed

Doing, Construing, and Discussing

Getting learners to do tasks in mathematics lessons is not sufficient to ensure that they make mathematical sense of what they are doing: it is often possible to follow a worked example as a template without having the faintest idea what it is all about, as many learners can attest. This fact is captured by Guy Brousseau in his notion of the *didactic contract* and in the associated *didactic tension*. The contract is largely implicit and based on the reciprocal responsibilities learners and teachers require of each other (Brousseau 1997, p 31). Learners expect that if they work at the tasks which they are assigned by the teacher, then somehow the required learning will take place. This leads to an endemic tension, for the more clearly and explicitly the teacher specifies the specific behaviour being sought in and by the task, the easier it is for learners to display that behaviour without actually generating it meaningfully from themselves, that is, without actually comprehending, understanding, or learning.

Getting learners to do tasks and also to discuss what they are doing is not enough to ensure learning: it is often the case that talking obscures doing (when the talk is a distraction, or when the talk is a displacement for getting down and doing something), and that talking rarely leaves a lasting impression. This is acknowledged in the framework Do-Talk-Record (Floyd *et al* 1981; see also Mason and Johnston-Wilder 2004a) which can act as a reminder, when preparing for and when conducting lessons, that each aspect contributes to the others, that it is wise to give learners tasks which involve them in trying to articulate what they are doing and what they are noticing when they are doing it, and that attempting to record too quickly can obstruct rather than support progress. Collaborative and cooperative working between learners is no panacea. There is no guarantee that meaningful learning of the intended subject matter is taking place. That said, many teachers are surprised to learn how ‘on-task’ most learner discussion is when tasks have been designed to provoke and promote mathematical sense-making.

Despite the desires of curriculum designers, textbook authors and teachers, there is no way to guarantee, ensure or force learning. Trying to ‘cause’ learning more often leads to tears than to success. Important influences in the socio-cultural and cognitive milieu of learners include the teacher’s personal commitment to mathematics and to learners, and their social, and particularly, mathematical ~~being~~ (in the sense of Heidegger, who crossed the word out as a reminder that it was not a ‘thing’ to be pointed to). Thus it is vital to be mathematical both with, and in front of learners in order that they experience mathematical practices.

Put another way, cause-and-effect is too simplistic a mechanism to account for the functioning of the human psyche. One reason for this is that human beings have the power to direct their attention, yet do not always exercise that control; they have the power to harness their energies, but do not always exercise that power. An important aspect of schooling is to provide conditions and experiences through which and in which learners discover that they can make choices, control the focus of their attention, harness their energies, and develop personal discipline.

One manifestation of these observations is that task and activity are not the same thing (Christiansen and Walther 1986): tasks are what teachers set learners to do; activity (in the sense of school activity) is what happens when learners engage in those tasks. But the tasks the learners ‘do’ are not always the task envisioned by the originator, nor always the task intended by the teacher who assigns them, since people tend either to do what they can from what they construe (Brown and van Lehn 1980) or else to wait until they are told exactly what to do (‘Is it an add or a multiply, Miss?’, Brown and Kucheman 1976). What matters is what learners are attending to, and how.

What is attention?

Both ancient and modern writers often connect attention with will, for attention is often, though not always, the subject, perhaps even the manifestation of will. However, I do not always exercise control over my attention: a sharp or loud sound, say of someone coming into the room, is enough to cause me to look up, presumably as an evolutionary residue from hunting and gathering. The fact that attention can be directed but is also subject to habit is what makes it crucial for learning and hence for teaching.

Attention is not a thing, at least in the sense of some thing to which you can point. I see attention not just as what puts me in touch with the world of my experience, but what creates and maintains that world. The totality of what I experience at any one moment *is* my attention. This is meant to include things of which I am subliminally or covertly aware, sometimes through body awareness, sometimes through social awareness, sometimes through emotional resonance, and sometimes through cognitive awareness. None of these need be conscious. For example, the psychoanalytic literature contains many attempts to describe or indicate how it is that an analyst listens through the words offered by the patient to something behind or between those words (see for example Bollas, 1987), not through conscious analysis, but through placing oneself in a sympathetic mode, in the sense of sympathetic vibrations in physics such as when humming near a piano makes some strings vibrate.

The old adage 'if you want to know about water, don't ask a fish' is highly pertinent, since in a strong sense 'I am where my attention is', or even 'I (cognitively) am my attention'. William James (1890, p 402) expressed it as "my experience is what I agree to attend to", although this articulation implies voluntary agreement, which may not always be the case. At each moment, as my attention shifts, the 'I' that inserts itself as the subject of predicates is the totality of that attention. Thus it is not very helpful to ask people direct questions about their attention.

Method of enquiry

Because my method of enquiry is perhaps somewhat unusual, at least when stated explicitly and conducted in public, a brief mention of the principal components may be helpful. The data I am offering does not consist of observations of learners in classroom, nor of teachers teaching; it does not even consist of reports of people's observations of their experience. The data I am offering consists of your own experience. In particular there will be experiences which resonate with and are brought to mind through immediate experience, and also experiences which challenge through the proposal of possible counter-examples or disbelief. To me this is the primary data arising from any qualitative and many quantitative studies: not what is presented, but what is stimulated in the readers who find their past experience brought to mind in resonance with current experience, enabling future action to be informed (Mason 1998, 2001).

What colleagues offer each other are distinctions which they find useful, and which they think others may find useful. Distinctions permit discerning of differences against a background of invariance or sameness, and lead to noticing in the future. Papers present evidence of various kinds as justification for the efficacy of proposed distinctions, but the test of any distinction is whether it illuminates past experience by helping to make sense of it, and-or if it informs future practice. A distinction coming to mind at some future moment may open the way to choosing to act in some way which is fresh and non-habitual, informed by the distinction.

If a distinction does not illuminate past experience, if it does not fit with or challenge past experience, then it may be that the distinction is either not relevant to you at the present time, for example because your concern is on other matters and your attention focused elsewhere, and it may mean that the tasks used as vehicles for generating experiential data need honing or replacing. It may also be that the distinctions being made are idiosyncratic and self-delusive. But then I maintain that these possibilities apply to most educational research!

Data generation

Having indicated the phenomena of interest and concern, it is vital to present some data. The data I am offering consists of your past experiences brought to the surface by present experiences through engaging in the task-exercises in this section.

Prelude

I begin by offering you some experiences.

Task: Selectivity

Read the message in **this type** *starting with the word* **Attention**.

There **Attention** *are* **is** *two* **partly** *sentences* **under** *intertwined* **your** *in* **control** *this* **but** *paragraph* **partly**. *Yet* **not**. *it* **In** *is* **reading** *possible* **mathematics** *to* **attention** *read* **can** *one* **be** *without* **attracted** *being* **but** *severely* **it** *distracted* **can** *by* **also** *the* **be** *other*, **blocked** *even* **out** *when* **by** *there* **unfamiliar** *are* **or** *familiar* **overly** *trigger* **complicated** *words* **formulae**. *like* **Reading** *mathematics*. **mathematical** *Even* **diagrams** *numbers* **and** *like* **symbols** *2004* **requires** *or* **that** *formulae* **attention** *like* **be** $2\pi r^2$ **differently** *can* **structured** *be* **at** *selected* **different** *against* **times** *as* **as** *long* **I** *as* **hope** *there* **to** *is* **demonstrate** *a* **in** *distinguishing* **this** *feature* **paper**.

Comment

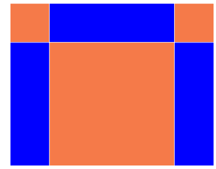
Did you try reading the italic non-bold text as well?

It is possible to select between bold ordinary and non-bold italic script with only the occasional distraction, and you can get better at it with practice. I had the sense in the lecture that most people were able to 'read' the two intertwined texts with only a few deviations such as when the word 'mathematics' appears twice in succession, once in each text, or when a number of formula appeared in the other text. Many experiments have been done to explore the dimensions of possible variation influencing selectivity of attention. This particular type of experiment is reported in Lechner (webref 2004)

Most general psychology texts treat attention *as* selectivity. The aim of this paper is to show that selectivity is only one of the dimensions of the structure of attention when it comes to mathematics.

Educators have recognised the importance of promoting the development of control and direction over attention with young children, but, except for Dina van Hiele-Geldof and Pierre van Hiele, perhaps not fully appreciated it as something which develops with use, and which plays a central role in learning and doing mathematics at all ages. For example,

Task: Counting squares and rectangles



The light-shaded shapes are squares.

How many squares can you count in the figure?
There are more than 3.
How many rectangles can you count in the figure?
There are more than 10.

This is a revision puzzle from a 1933 textbook for novice teachers (now mislaid!)

Comment

What did you do with your attention in order to locate more than 3 squares, and more than 10 rectangles? How did you convince yourself that you had them all?

Finding a way to be systematic so that you can be confident you have them all involves breaking the task into sub-problems (how many rectangles in the top 'row'?) and seeing that it will be the same as the number in the bottom 'row' and in the two rows combined. Paying attention to how you see often reveals a generality: seeing it in terms of rows enables you, with care, to answer the question for three and more rows.

It is well known that in order to make sense of geometrical figures you need to discern details, to locate relationships, and to perceive relationships as properties that other objects might share, as you did in the rectangle counting. The next task suggests that arrays of numbers can act as a similar bridge into the domain of number.

Task: Odd table

What relationships can you detect in this array? Don't be satisfied with just one or two!

1	3	7	13	21	31	43	57	…
	5	9	15	23	33	45	…	
		11	17	25	35	47	…	
			19	27	37	49	…	
				29	39	51	…	
					41	53	…	
						55	…	

Comment

You probably noticed the triangularity, and the sequence of odd numbers. Did you notice the locations of the square numbers (including the virtual presence of the even squares in the middle of every other column)? Did you notice that the columns add up to cubes (and that there are easy and hard ways to add up the columns!)? Or that in the first row 3 divides the second, fifth and eighth terms, while in the second row, 5 divides the first and sixth terms? Did you find yourself wondering if any of the things you noticed carry on as the sequence is extended.

This array is also known as the Fibonacci Triangle (Ollerton and Shannon, 2003). Mention of the word Fibonacci probably sent you back to the table to try to see what it has to do with the Fibonacci sequence. What does that say about your attention?

Statement of the theme

It is perhaps time now to present the central idea, the main theme of this paper, first through a task, then in words in the commentary, and then in the following subsection, an articulation independent of particular tasks.

Task: Blinded by symbols

How does your attention alter as you contemplate the following expression?

$$\frac{(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)}A+\frac{(x-a)(x-c)(x-d)}{(b-a)(b-c)(b-d)}B+\frac{(x-a)(x-b)(x-d)}{(c-a)(c-b)(c-d)}C+\frac{(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)}D$$

Comment

You might find yourself overwhelmed by all the letters and complexity. You might then become aware of the horizontal bars, the plus signs, and the capital letters. This gives access to the fractions each of which is a product involving x in the numerator and some lower case letters in the denominator. Using the power to focus selectively, you might be drawn to, say, the first fraction, and to observe things that are the same about numerator and denominator, and also things that are different. This could lead to noticing that where in the numerator there is an x, in the denominator there is an a. To do this requires a matching process which can be holistic or can be detailed, term by term. Note that controlling visual attention in this way is not as easy for some people as for others, particularly those with strong forms of dyslexia. Once the relationship between the role of x and the role of a is detected, the first fraction is seen to have a property: when x is a, the fraction simplifies to 1. But to think in these terms already requires sensitivity to the importance of the number 1, and to experience with substitution. Awareness of 1 may trigger thoughts about 0, and the property that when x is b, c, or d, the fraction value is 0. Interest is aroused. You turn to the other fractions and see that they are similar. So this whole expression has the value A when x is a, B when x is b, C when x is c, and D when x is d. Remarkable!

You may have found, as one reviewer did, that your intention influenced your attention. The intention to simplify or recast the expression structures attention differently to the intention to make sense of it as it stands.

In fact, this is the expression for a cubic polynomial going through four specified points (a, A), (b, B), (c, C), and (d, D).

Until you discern details you cannot do anything but be aware of the whole. This ‘whole’ may shift from the entire expression to one of the four terms added together, but the experience is of ‘stuff’, of ‘a mess of symbols’, as yet undifferentiated. Some learners are so taken by all the detailed symbols that they find it difficult to gaze, to ‘fuzz’ some details in order to get a sense of the overall structure, here, of four terms added together. It is also possible to become ‘fixated’ by the occurrence of certain symbols such as the x’s.

As you focus on a detail, discerning it from what is around it (foregrounding against a background, which requires stressing some features and consequently ignoring others) you become aware of relationships. Here, there are relationships within a part such as one of the fractions (comparing numerator and denominator, seeing the distinct role of one of the lower case letters and the corresponding capital), and also between the parts (each has a capital letter and a fraction with four terms in both numerator and denominator, …). Human beings naturally seek out similarities and recognise relationships. That is how sense is made of what our senses bring us. Note however that there is tremendous selectivity long before there is any cognitive processing: you have probably not paid attention to the font used, nor to the density of the black against the white on the page, nor the feel of the paper on your fingers.

Having analysed each of the terms (fraction with capital) and located some similarities between them, it is possible to change the way you are thinking and to ask, for example, whether all the possible combinations of triples from *a*, *b*, *c* and , are present. This is a step towards property-making or 'proposing properties': isolating a relationship or a similarity and then looking to see if other objects have that same relationship. For example, you might think to ask what the corresponding formula would look like for just two points, or for three or five points. To write them down you would be using incipient properties. Articulating those properties turns them into a something which can be used elsewhere. For two points, there would be two values *A* and *B*, with corresponding *a* and *b*. If you see the fractions as constructed so as to have the value 1 when *x* is the repeated value in the denominator, then you can use this property to manufacture a fraction for any number of specified values:

$\frac{x-b}{a-b}A$ as one of the two terms needed for two points,

and $\frac{(x-b)(x-c)}{(a-b)(a-c)}A$ for one of the three terms needed for three points,

with clear generalization to other numbers of points. Thinking in terms of properties makes it easier to make deductions.

Definitions in mathematics arise when a shift is made from objects being tested to see if they satisfy properties, to deductions being made starting just from knowing that an object has those properties. Thus if there is an expression which takes the value 0 at $x = b$, c and d, and the value 1 at $x = a$, then versions of it can be assembled to make a formula for a polynomial passing through four points.

As another example, take a grid of number facts presented to learners.

Task: Sequential generalisation

$2(2 \cdot 3 \cdot 3 + 3 + 3) + 1$ $= (2 \cdot 3 + 1) \cdot (2 \cdot 3 + 1)$	$2(2 \oplus 4 \oplus 3 + 4 + 3) + 1$ $= (2 \oplus 4 + 1) \oplus (2 \oplus 3 + 1)$	$2(2 \oplus 5 \oplus 3 + 5 + 3) + 1$ $= (2 \oplus 5 + 1) \oplus (2 \oplus 3 + 1)$
$2(2 \oplus 3 \oplus 4 + 3 + 4) + 1$ $= (2 \oplus 3 + 1) \oplus (2 \oplus 4 + 1)$	$2(2 \oplus 4 \oplus 4 + 4 + 4) + 1$ $= (2 \oplus 4 + 1) \oplus (2 \oplus 4 + 1)$	$2(2 \oplus 5 \oplus 4 + 5 + 4) + 1$ $= (2 \oplus 5 + 1) \oplus (2 \oplus 4 + 1)$
$2(2 \oplus 3 \oplus 5 + 3 + 5) + 1$ $= (2 \oplus 3 + 1) \oplus (2 \oplus 5 + 1)$	$2(2 \oplus 4 \oplus 5 + 4 + 5) + 1$ $= (2 \oplus 4 + 1) \oplus (2 \oplus 5 + 1)$	$2(2 \oplus 5 \oplus 5 + 5 + 5) + 1$ $= (2 \oplus 5 + 1) \oplus (2 \oplus 5 + 1)$
$2(2 \oplus 3 \oplus 6 + 3 + 6) + 1$ $= (2 \oplus 3 + 1) \oplus (2 \oplus 6 + 1)$	$2(2 \oplus 4 \oplus 6 + 3 + 6) + 1$ $= (2 \oplus 4 + 1) \oplus (2 \oplus 6 + 1)$	$2(2 \oplus 5 \oplus 6 + 5 + 6) + 1$ $= (2 \oplus 5 + 1) \oplus (2 \oplus 6 + 1)$

Are these statements true? Did you check?

What entry would you expect to see in the fifth column and sixth row if the patterns were to continued? Generalise.

Comment

At first it seems like a mass of numbers, signs for times and plus, and 2s. The profusion of 3s in the upper left cell may be off-putting if not confusing, making it hard to discern any detail.

Almost immediately the equals sign is seen to separate two expressions, one on each line (sub-objects discerned). It becomes possible then to focus on one expression and to calculate its value. Or what might be striking is the similarity between lines, in different cells in each column. Focusing on what is changing line by line might suggest a way to extend beyond the grid shown, and having continued one or two more cells, awareness of structure might lead to a sense of properties or format of each line in each cell. This property-awareness not only facilitates but constitutes expressing a generality. Asking what else might change, or deciding where you would expect to find

$$2(2 \cdot 7 \cdot 9 + 7 + 9) + 1 = (2 \cdot 7 + 1) \cdot (2 \cdot 9 + 1)$$

is likely to expand the sense of form of a cell, to broaden the sense of relationships, to suggest a more general property, and to emerge as an expression of generality. Validating conjectures requires reasoning solely on the basis of arithmetical properties.

The structure of this task can be used in many different topics and settings. At a more sophisticated level, consider the following

	$2(2ab + a + b) + 1$ $= (2a + 1)(2b + 1)$		$2(2ab + 3(a + b)) + 9$ $= (2a + 3)(2b + 3)$	
	$3(3ab + a + b) + 1$ $= (3a + 1)(3b + 1)$	$3(3ab + 2(a + b)) + 4$ $= (3a + 2)(3b + 2)$		
	$4(4ab + a + b) + 1$ $= (4a + 1)(4b + 1)$		$4(4ab + 3(a + b)) + 9$ $= (4a + 3)(4b + 3)$	

Again the profusion of symbols may be off-putting at first, but after a moment or two discerning of leading coefficients in the first column, perhaps, or discerning the presence of the equals signs, leads to recognition of form or structure. The presence of blank cells suggests an invitation to fill them in, and to do so requires explicit reproduction of the sensed structure, in the form of relationships as properties. Using the first column, perhaps with a strategy such as getting learners to “say what you see”, or to “watch what you do” as you copy what is invariant and change what is changing, provides more explicit access to structure. Moving to other cells is already signalling an implicit sense of property. Expressing a generalisation about the contents of *any* cell leads to seeing the structure as a starting point for making deductions: to test whether a given number can be expressed in the form $mab + n(a + b)$ for integer values of a and b, instead of lots of try and improve, you multiply the given number by m, add n^2, then factor it in all possible ways, looking for two factors which are both of the form $mx + n$. Here the reasoning has to be solely on the basis of the form of the numbers, not on any particular numbers.

Task: Magic square facts

> Imagine someone has constructed a 3 x 3 magic square, but you don’t know the entries. However, you do know that, for example, the sum of any two entries in the same row is equal to the sum of the two other entries in the column corresponding to the third entry of that row. What other such relationships can you find? Try the same for 4 x 4 magic squares and larger.

Comment

Apart from the work needed to interpret the claim, the main point is that you can reason on the basis of the one property of magic squares: that the row, column and main diagonal sums are all equal. From this alone you can deduce all sorts of other sums which must be the same. There is no need to have an example to hand on which to ‘check’ results, which might in fact get in the way of the reasoning. Everything follows solely from known properties. The idea for this came from Johnny Baker (private communication) doing the same with magic hexagons.

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This task provides quintessential experience of reasoning on the basis of properties alone, without recourse to particular examples for inspiration or checking. Indeed, particular instances are more likely to intrude and confuse, to direct attention away from the reasoning rather than supporting and inspiring it. In most mathematical topics however, learners have recourse to particulars and to several properties, so that it is difficult to isolate the defining properties and to use them alone as the basis for reasoning. Perhaps this is one of the reasons why teaching proof is so difficult in mathematics. There are evident difficulties for some learners simply with reasoning. More problematic still is the shift away from accumulating knowledge of various properties a specific object has or might have, to isolating properties which define and hence characterise those objects. To reason on the basis of specified or defining properties alone requires back-grounding all other information, facts and properties of which you are aware so that reasoning can proceed 'axiomatically'.

Theme elaborated

The basic idea is that attention shifts rapidly between holistic encompassing (gazing), discerning distinctions (stressing and ignoring, foregrounding and back-grounding), recognising relationships amongst discerned features, perceiving properties that objects in general may possess, and reasoning based on deducing from definitions (selected properties) and axioms. Whether attention is the subjective experience of physiological functioning, as Théodule Ribot (1890) would have it, or the engine for physiological response to environment, as William James (1890) suggests, reflection on the experience of working on tasks such as those presented earlier suggests to me quite distinctive if subtly different forms of attention:

Holding Wholes (gazing)

Discerning Details (features and attributes)

Recognising Relationships (part-part, part-whole)

Perceiving Properties (leading to generalisation)

Deducing from Definitions (axioms etc. stated independently of particular objects)

Shifts between these are rapid, often subtle, but vital in order to engage in mathematical thinking. While gazing, some sudden movement, perhaps even apparent motion produced from circadian eye movement can suddenly switch attention to awareness of details amongst a mass of other, undiscerned detail. As details are detected and discriminated, the mind automatically looks for relationships: differences and samenesses. To do this requires something being relatively invariant as a background against which to detect change. Recognising relationships tends to focus on particulars, whereas perceiving properties is a move to the more general, to the particular as exemplary or paradigmatic. Formalising in mathematics is the overt action which accompanies a shift from perceiving properties to taking certain properties as definitive and so as the basis for reasoning.

If learner attention and teacher attention are sometimes significantly differently structured, then confusion is the most likely outcome. More particularly,

if some learners are attending holistically when the teacher is discerning specific details;

if some learners are discerning details when the teacher is talking about relationships amongst details;

if some learners are recognising relationships amongst discerned elements when the teacher is talking about properties of objects in general;

or if some learners are thinking about properties when the teacher is deducing from properties;

then here is likely to be a mismatch, a failure of communication.

Reprise da capo
Distinctions such as those being proposed between different forms or structures of attention are only useful if they help make sense of past experience, and-or inform future practice. The following notes suggest some ways in which the structure of attention might have been experienced in the earlier tasks.

Selectivity task: your perceptual mechanisms were able to discern the bold type and to foreground it, pushing the italic text into the background, with more or less success. Foregrounding and back-grounding is a form of selectivity, which is what psychologists see as the main function of attention.

Counting squares task: discerning squares requires maintaining the relationships which characterise squares as properties, but at the same time or perhaps rapidly interpolated, attention needs to hold onto some shapes already counted while looking for ones not yet counted. Finding a systematic way to locate all the squares essentially involves reasoning on the basis of properties of squares in order not to double count. The same happens in a slightly more complex fashion for counting rectangles.

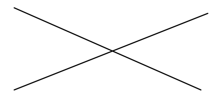
Odd table task: having discerned some simple facts (triangularity, odd numbers) it is natural to assume that these relationships will continue as the array is extended. In other words, you naturally turn relationships into properties. To verify other relationships which are conjectured to continue as properties requires reasoning on the basis of specified properties (odd numbers and triangularity) alone.

The conjecture being advanced is that these various structures in the form of attention are not restricted to sophisticated adults, but rather form the experience of the youngest of children. In mathematics lessons, apparent failure to communicate or to ‘be heard’, apparent failure to comprehend what is being said and done, may be due to momentary differences in the structure of attention rather than to ‘abilities’. Since learners’ self-image and self-confidence are often delicate, mismatches in what is being attended to and how may have serious consequences. Cultural expectation and peer pressure are additional forces, and together these can open a chasm between teacher attention and learner attention which may take considerable effort to re-bridge. There are evident connections with van Hiele levels, with the onion model of understanding, and with the SOLO taxonomy which are elaborated in section 3.

Variations
In this section I have space to offer only one more example in which the fluid nature of the structure of attention can be experienced. My claim is that sensitivity to the structure of attention is relevant to all aspects of teaching mathematics, at every level.

A classroom example
My wife, Anne Watson, drew my attention to a phenomenon observed in many classrooms concerning ‘vertically opposite angles’ (when two straight lines meet at a point,

there are two pairs of equal angles, referred to in English as ‘vertically opposite’). When the teacher says ‘vertically opposite angles’, learners are often able to complete the sentence with ‘are equal’. But it turns out that sometimes this is only a memorised mantram. When given a diagram such as the ones displayed here,



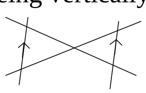
some learners see only two lines crossing;

some learners discern at most one pair of angles (usually the ones ‘at the top and the bottom’ of the intersection), but do not see them as being related (as equal);

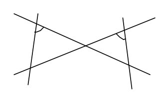
some learners discern one pair of opposite angles and see them as equal, perhaps through seeing them as generated by rotating one line about the intersection;

some learners are aware of the property of ‘being vertically opposite’ as implying equality, and experience and instantaneous trigger to see the two angles as ‘being equal’, yet don’t use these equivalences to carry out reasoning;

some learners are able to reason about the equality of angles because of being vertically opposite, in situations such as the following:



Show that the two triangles are similar (have equal angles).



A good way to prompt learners to reveal the extent of their awareness and the structure of their attention is to ask them to construct diagrams which display a pair of vertically opposite angles, or which use a pair of vertically opposite angles to reason about the equality of a pair of apparently unrelated angles. Asking for another example, and then another example often invokes creativity and variations of which learners are becoming aware (Watson and Mason 2002, 2004).

One difficulty may be that the words ‘vertically opposite’ bring meaning which is not actually wanted mathematically: the ‘verticality’ is irrelevant, as sometimes the angles are horizontally related rather than vertical. Some learners may even be reinforced in their view that mathematics has no meaning and consists simply of techniques for getting answers. This illustrates perfectly the effects of teacher and learner attention being differently structured.

What are learners attending to?

I hope that these experiences have raised the question not only of what learners are actually attending to in lessons, but how they are attending to it. That is, how can you detect when a learner is attending holistically, discerning details, recognising relationships, perceiving properties, or ready to reason on the basis of properties alone? Since these types of attention are often fleeting, it is not so much a matter of capturing one, but developing strategies for directing or focusing attention in a way which is pertinent to the learning, while at the same time being sensitive to different learners’ needs to dwell in a particular form of attention before being rushed on to another.

Theme (Reprieve)

The upshot of these examples is an illustration of how human attention shifts rapidly between different ways of attending and different foci of attention. Sometimes there are several of these going on either simultaneously, or in swift succession. Attention is the basis for ontological acts that is for the creation of entities, of objects. There is

> awareness of or focus on wholeness, a holistic encompassing, such as *gazing*;
>
> awareness of or focus on discerning 'this not that', and so literally creating objects and sub-objects;
>
> awareness of relationships or similarities between features comprising sub-objects at any level of detail;
>
> awareness of or focus on properties as attributes that (sub-)objects might satisfy or possess;
>
> awareness of or focus on reasoning solely on the basis of properties which requires specified properties to be seen as definitions or as axioms, not just as attributes.

The suggestion is that shifts between these states, that is, shifts in the structure of attention, are going on all the time, though the property-making to some extent and the defining and axiomatising particularly are perhaps predominately mathematical activities. By being aware of these shifts in your own attention, you can be more alert to the possibility that learners are not making the same shifts with you, and that there are things you can do to try to prompt those shifts. For example, if you notice that you are thinking in terms of properties, you can pause and be explicit that that is what you are doing, or you can pause and give learners time to articulate the relationships they recognise, then offer other objects to see if these have the same relationships, before prompting a move to articulating the relationship as a property.

Learning from the past

There are evident close similarities with van Hiele levels in particular but also notable contrasts with approaches taken to the nature of attention in the psychological literature. I begin this section with similarities before moving on to differences.

Similar approaches

Within mathematics education literature there is very little on attention as such. One notable exception is a paper by Kaye Owens and Ken Clements (1998). They found that obstacles encountered in problem solving could often be accounted for by selective focus of learners' attention on particular details. They arrive at the core role of attention as the basis for selectivity. Attention is seen as co-emergent with responsiveness to stimuli from the teacher and from fellow learners. They also found that

> "changes in attention often heralded changes in understanding" (p 213),

where I would say that attention changes *constitute* changes in understanding. If the changes are temporary, so is the understanding; if the changes inform or influence future behaviour then change in understanding is more robust.

Walter Whiteley (private communication) has worked long and hard to convince people that geometry is not about proof but about working with mental imagery, and that geometry most naturally begins with topological awareness and three dimensional space, not Euclidean geometry and two dimensional space. He points out that the van Hiele levels as originally conceived were driven by a 2D Euclidean view rather than 3D topological experience. He also notes that ‘students do not see what we see - unless they apprentice with … images and … reasoning’, which is consonant with and explained by distinguishing different forms or structures of attention.

Attention and understanding

Although attention does not feature prominently in mathematics education research currently, there is a great deal of writing about what constitutes understanding. From my perspective understanding, whatever it means, has to do with how attention is structured. Indeed, one of my long-standing conjectures is that

> each mathematical term signals that someone experienced a shift in the way they perceived, in what they noticed, in how they attended, and that in order to integrate that term and its use into your own thinking it is necessary to experience a corresponding shift in your own attention.

I say ‘corresponding’ because differences in experience and culture may mean that the actual shift experienced is different. In this section I consider some evident similarities between the structures of attention I have identified and those of some other authors.

Anyone familiar with van Hiele levels (van Hiele-Geldof 1957, Burger & Shaunessy 1986) will be aware of close similarities with the structures of attention being proposed. Here is one version (based on Burger & Schaunessy *op cit*), to which I have appended a version cast in terms of what reasoning might look like:

Level 1: Visualization (reasoning based on direct perception)

Level 2: Analysis (reasoning based on relating component parts and attributes)

Level 3: Abstraction (reasoning based on necessary conditions as known facts)

Level 4: Informal Deduction (reasoning based on relating properties)

Level 5: Formal Deduction (reasoning from axioms systematically)

Pierre van Hiele generalised these beyond geometry (van Hiele, 1986) but in the process made them even more abstract and for me harder to connect to moment-by-moment experience, which is where I think it is important to focus in order to influence and improve learners’ experiences of being taught mathematics. Although I was familiar with the van Hiele levels, I had found them difficult to work with, partly because I am unconvinced by attempts to specify distinct levels. I came to the structures of attention through an entirely different route based on Eastern sources (Bennett 1956-1966).

The difference between the structures identified here and the van Hiele levels lies precisely in the notion of levels. Rather than seeing these structures as levels or even as hierarchical qualities in the way researchers have developed the van Hiele ideas to date, I am proposing the radical stance that these so-called levels are actually descriptions of the way that people attend all the time, often with rapid shifts from one to another.

Susan Pirie and Tom Kieren developed an onion-skin model of understanding (Pirie and Kieren 1989, Pirie and Kieren, 1992, Kieren 1994) which they have used to capture the way in which the manifested form of understanding shifts back and forth between layers, as depicted below. Flashes of ‘understanding’ are of many different types, as their model displays, and each of these is a form or focus of attention.

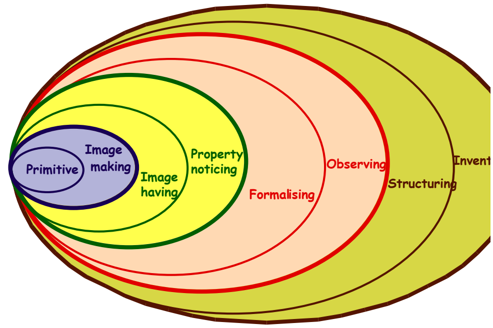


Image-making depends upon discerning and discriminating, and image-having contributes to seeking relationships. The slip from relationship between particulars to relationship as property is perhaps not always as immediate in mathematics classrooms as it is in ordinary life, perhaps because teachers tend to skip over it as being obvious and automatic. Furthermore, property-noticing and property-making is an essential step on the way to formalising, which requires separating a property as simply one attribute of an object, to property as the basis for further reasoning.

A compatible but non-mathematical approach to the subject of attention can be found in Pierre Lacout (1969) who writes that “contemplative silence is a special form of attention” (p 7), referring to holistic gazing rather than analytic discernment as a form of prayer.

Different approaches

In his comprehensive, and to my mind still unsurpassed treatise on *The Principles of Psychology*, William James (1890) devotes an entire chapter to the subject of attention. However, the difficulty of defining *attention*, and the shift to a behaviourist perspective which eschewed internal workings of the brain or psyche, meant that psychologists moved away from attention as a subject for enquiry for a time. In the 1960s behaviour-dominated researchers studying animals began to realise that there was more to animal and human interaction than merely the triggering of specific behaviour patterns. A moment’s recollection of what it is like when you are talking or listening to someone who never looks at you suggests that the admonition ‘pay attention!’ signals a truth about human and animal psyche: attention is something which people and other animals possess: it can be directed, but is often directed for us, and most people like to receive attention from others, at least in some circumstances. There has of course been considerable interest in attention-seeking behaviour by children and adolescents, which contributes a socio-psychological dimension to the structure of attention. More pertinently, it is possible not only to be aware of other people’s attention, but even to become more sensitive to its nature and form.

Attention and construction

For William James, attention is what makes it possible to perceive, conceive, distinguish and remember, in short, the basis of all our psychological functioning (James, 1890 p 424). As might be expected, he deals with a number of important issues concerning attention in general. For example, he argues on the basis of Helmholtz’s experiments that attention is not simply what the eyes are looking at, or indeed any other particular source of sense impressions (p 438). He links attention to anticipative imagination (p 439-411) as a prerequisite for discerning anything at all.

James develops this theme of discernment, or discrimination, to make use of what he calls Helmhotz's law, that

> "we leave all impressions unnoticed which are valueless to us as signs by which to discriminate things" (p 456).

James then goes on to discuss pedagogic implications such as that it is useful for teachers to work with learners to strengthen and attract their attention in order to improve motivation, since people engage with what catches their attention (James, 1890, p 446). To do this requires being aware of what in learners' previous experience can be used as a basis of previous attention-experience, what John Dewey referred to as 'psychologising the subject matter' (Dewey, 1902, p 12).

Although convinced that attention is the cause rather than the result of sense impressions, James gives a sensitive exposition of the two sides of the argument (p 448). He sees attention as a form of 'free energy', since when you make an effort to attend to something you can sustain it for only very short periods before attention wanders (p 420) requiring a further expenditure of effort, but when attention is engaged it requires no energy expenditure at all for it to remain focused for long periods of time. This observation was also made and exploited by P. D. Ouspensky (1950).

Where I differ with James is in his metaphor of attention or consciousness as a flowing stream, for it seems to me that his own descriptions (e.g. James, 1890 p456 quoting Müller), as well as my observations, lead to the conclusion that attention is suddenly sharp and alert, and then slowly declines into absence of awareness until some fresh stimulus wakes it up again. The sense that we have of experience flowing by is actually much more episodic, as attempts to reconstruct recent and distant experiences demonstrates all too clearly.

Preparing for the future

The only virtue in making distinctions is to inform future practice and to facilitate communication. Distinctions may be considered to comprise or contribute to a theory, a framework, or a perspective, but each of these is ultimately aimed at enabling people to improve their practice through noticing finer distinctions than previously. That practice might be to do with analysing classroom incidents, but more positively, it can mean having fresh choices open up both while preparing and in the midst of the moment by moment flow of interacting with learners. The conjecture being put forward is that distinguishing different forms of attention can open up a range of possibilities which might otherwise be invisible.

For example, the different structures of attention proposed offer opportunities to be sensitive to the various ways in which learners are attending, making it possible to choose to dwell longer so that learners can gain confidence before being expected to attend differently, or to direct or attract attention suitably.

Hans Freudenthal (1991) drew attention to the difficulty of getting learners to move away from 'it just is so' as justification for a conjecture, to 'it is so because …' and reasoning on the basis of properties. Even the recognition that an assertion is about a relationship holding and constituting a property involves 'pre-reasoning', and is essential in order to manipulate properties.

A plausible conjecture is that reasoning involves a shift of attention away from discerning details, recognising relationships and perceiving properties, to isolating some of those properties as the basis for reasoning. To let go of 'all the things you know about something' and to start reasoning simply on the basis of some stated properties is quite sophisticated, certainly in historical terms, as well as in developmental terms. This is not to say that reasoning itself arises late in development. Very young children display moments of reasoning from a property to draw a conclusion. What is required mathematically is to undertake that shift intentionally and knowingly. This is most effectively achieved by drawing learners' attention to what they have done spontaneously, in order to support them in choosing to do it another time in the future.

Summary

The basic idea in this paper is that attention shifts between holistic encompassing, discerning distinctions (stressing and ignoring, foregrounding and backgrounding), recognising relationships amongst discerned features, perceiving properties that objects or elements may possess, and deducing from definitions and axioms. These shifts can be rapid, but can be blocked; they are often subtle, but always vital to learning. They have been illustrated through your experience of various task-exercises that I have offered, in the hope that you recognise something of what I myself have noticed, and that you also recognise similar phenomena in other experiences of your own.

I am trying to understand deeply the nature of my own attention in the context of mathematics with a view to informing my own practice when working myself, and with others on mathematics. I am not offering a mechanism to try to ensure or cause a particular form of attention, which I consider would contradict the way learning takes place. Nor am I offering a 'model' of how attention can or should be structured.

The importance for me of being aware of different attention structures and of shifts between them is to alert myself to situations where there is a mismatch, or where someone has become stuck in one structure.

Bibliography

Bennett, J. (1956-1966). *The Dramatic Universe* (four volumes), London: Hodder & Stoughton.

Bollas, C. (1987). *The Shadow of the Object: psychoanalysis of the unknown thought,* London : Free Association Books.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics: Didactiques des Mathématiques, 1970-1990,* Balacheff, N. Cooper, M. Sutherland, R. and Warfield, V. (trans.), Kluwer, Dordrecht.

Brown, M. & Kuchemann D. (1976). 'Is it an add, miss?'. *Mathematics in Schools* **5**(5) p 15-17.

Brown, J. & van Lehn, K. (1980). *Repair Theory: a generative theory of bugs in procedural skills, Cognitive Science* **4,** p 379-426.

Burger, W. & Shaunessy, J. (1986). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education* **17**(1), p 31-48.

Christiansen B. and Walther, G. (1986). Task and Activity, in B. Christiansen, G. Howson & M. Otte, *Perspectives in Mathematics Education*, Dordrecht, Reidel, p 243-307.

Floyd, A., Burton, L., James, N., Mason, J. (1981). *EM235: Developing Mathematical Thinking* Open University Course. Milton Keynes: Open University.

Freudenthal, H. (1991). *Revisiting Mathematics Education*: *China lectures*, Kluwer, Dordrecht.

James, W. 1890 (reprint 1950), *The Principles of Psychology*, New York: Dover.

Kieren, T. (1994). Bonuses of Understanding Mathematical Understanding, in Robitaille, D. Wheeler, D. & Kieran, C. (Eds.) *Selected Lectures from the 7th International Congress on mathematical Education.* Quebec: Les Presses de l'Université Laval, p 211-228.

Lacout, P. 1969 (English version 1970), *God is Silence*, London: Friends Home Service Committee.

Lechner, M. (webref, accessed august 2004) search: Lechner, psychology, attention)

Mason, J. & Davis, J. (1989). The Inner Teacher, The Didactic Tension, and Shifts of Attention. In G. Vergnaud, Rogalski, & M. Artigue, (Eds), *Proceedings of PME XIII*, Paris, Vol 2 p 274-281.

Mason, J. (1989). Mathematical Abstraction Seen as a Delicate Shift of Attention, *For the Learning of Mathematics* 9 (2) p2-8. Mason, J. (1989). Mathematical Abstraction Seen as a Delicate Shift of Attention, *For the Learning of Mathematics* **9** (2) p 2-8.

Mason, J. (1998). Enabling Teachers to be Real Teachers: necessary levels of awareness and structure of attention, *Journal of Mathematics Teacher Education* **1** (3) p 243-267.

Mason, J. (2001). Teaching for Flexibility in Mathematics: being aware of the structures of attention and intention, *Quaestiones Mathematicae*, 24 Suppl 1, p 1-15.

Mason, J. (2002). *Researching Your own Practice: The Discipline of Noticing*, London: RoutledgeFalmer.

Mason, J. (2003). Structure of Attention in the Learning of Mathematics, in J. Novotná (Ed.) *Proceedings, International Symposium on Elementary Mathematics Teaching*, p 9-16. Prague: Charles University.

Ollerton, R. & Shannon, A. (2003). Some Useful Integer Arrays for the teaching of Generalization Skills. *International Journal of Mathematical Education in Science and Technology* **34** (1) p 135-143.

Owens, K. & Clements, M. (1998). Representations in Spatial Problem Solving in the Classroom, *Journal of Mathematical Behavior* **17** (2) p 197-218.

Pirie, S. & Kieren, T. (1989). A Recursive Theory of Mathematical Understanding, *For the Learning of Mathematics* **9** (4) p 7-11.

Pirie, S. & Kieren, T. (1992). The answer determines the question – interventions and the growth of understanding. In W. Geeslin & K. Graham (Eds.) *Proceedings of PME XVI*, 2. New Hampshire, p 1-8.

Ribot, Th. (1890). *The Psychology of Attention*, Chicago: Open Court.

van Hiele, P. (1986). *Structure and Insight: a theory of mathematics education*, Developmental Psychology Series, London: Academic Press.

van Hiele-Geldof, D. (1957). The Didactiques of Geometry in the Lowest Class of Secondary School. In D. Fuys, D. Geddes, & R. Tichler (Eds.) 1984, *English Translation of Selected Writings of Dina van Hiele-Geldof and Pierre M van Hiele*, National Science Foundation, New York: Brooklyn College.

Watson, A. & Mason, J. (2002). Extending Example Spaces As A Learning/Teaching Strategy In Mathematics, in A. Cockburn & E. Nardi (Eds.) *Proceedings of PME 26*, University of East Anglia, Vol 4 p 377-385.

Watson, A. & Mason, J. (2004). *Mathematics as a Constructive Activity: the role of learner-generated examples*, Mahwah USA: Erlbaum.

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