Creating opportunities for students to reinvent mathematics

Koeno Gravemeijer
Freudenthal Institute and Department of Educational Research, Utrecht University, The Netherlands

Introduction

Reform mathematics education is often portrayed against the background of traditional instruction. Such a comparison usually leads to recommendations on what teachers should *not* do. This, however, begs the question what teachers *can* do to proactively support their students' mathematical development. This question is especially pressing, since both conventional instructional design theories, and conventional teaching approaches, do not have much to offer in this respect.

Reform mathematics education asks for a classroom culture that differs sharply from the so-called "transmission model of teaching". Teachers will have to establish a classroom culture in which whole-class discussions involving conjecturing, explaining, and justifying, play a crucial role. Crudely put, reform in mathematics education aims at shifting away from "teaching by telling", and replacing it by "students constructing", or "reinventing". This implies, however, a shift in emphasis from what *teachers* do, to what *students* do. So the problem arises of how to direct this process, or: How can we make students reinvent what we want them to reinvent?

Conventional instructional design strategies are not of much use for designing mathematics education that puts the students' own ideas and input at the forefront, as they are typically based on some form of task analysis. Task analysis produces a series of learning objectives that may make sense from the perspective of the expert, but not from the perspective of the learner. Moreover, there is no room for the personal input of the learner in a set up that is based on task analysis.

Still, I would argue that mathematics education aiming at capitalizing on the input of the students asks for thorough planning—although a different kind of planning. What is needed for reform mathematics education is a form of instructional design that supports instruction that helps students develop their current ways of reasoning into more sophisticated ways of mathematical reasoning. My claim is that an instructional approach in which the teacher builds proactively on students' contributions is only possible if the teacher has a resource of exemplary instructional activities at his/her disposal. These instructional activities should have the potential to give rise to a variety of interpretations and solutions, which can be used to advance the teacher's instructional agenda. Apart from exemplary instructional activities, from which teachers can choose, and which they can adapt to their needs, teachers also need a framework of reference to guide their educational decision-making.

Providing both is exactly the objective of what is currently called "design research". The goal of the type of design research I have in mind is to develop both prototypical instructional sequences consisting of series of (exemplary) instructional tasks, and local instruction theories that underpin those sequences, where the latter encompass both theories about the process of learning specific topics and theories about means designed to support that learning.

In this paper, I will capitalize on design research in realistic mathematics education (RME), carried out in the Netherlands and the USA, to elaborate on the instructional design heuristics constituting RME theory, in order to sketch a preliminary answer to the above issues of instructional design.

Common-sense mathematics

The problem with the transmission model of teaching is that, in general, the transmitted mathematics does not make sense to the students. In fact, we may observe that to many people mathematics does not make sense. This is reflected in jokes on mathematics education, such as the one in figure 1.

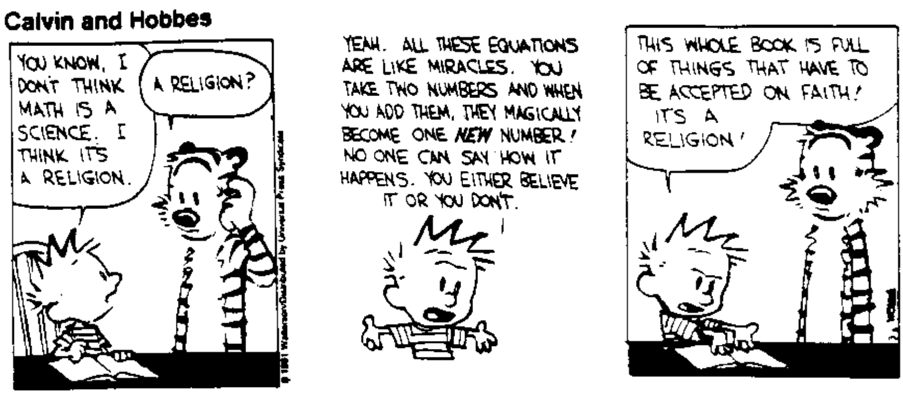

Figure 1. Math is a religion
(Copied from Bill Watson (1992), *Attack of the deranged mutant killer monster snow goons,* Kansas City: Andrews and McMeal, page 112.)

Freudenthal (1991) offers an alternative in arguing that mathematics education should start and stay within reality. This thesis is often interpreted as implying that mathematics education should only be concerned with applied problems in practical, or authentic, situations. That, however, is not what Freudenthal meant and that is also not what realistic mathematics education means. Freudenthal (1991) defines reality as, “What common sense experiences as real.” And he argues that what is common sense for a layman is different from what is common sense for a mathematician. We may connect this with an observation of Davis and Hersh (1986), who note that mathematicians talk among each other about mathematical objects—that are completely imaginary to the layman—as if they are real. And they go on to say that for these mathematicians, these object are real. We may, in sum, conclude that reality is something that develops.

Let me try to illustrate this development of mathematical reality with a simple example. For you as a reader, “1+1=2”, is common sense, but this may be different for young children. At a certain age, young children do not understand the question: “How much is 4+4?” Even though they may very well understand that 4 apples and 4 apples equals 8 apples. The explanation is that, for them, numbers are still tied to countable objects, like in “four apples.”

At a higher level: 4 will be associated with various number relations:

4 = 2+2 = 3+1 = 5-1 = 8:2 etc.

At this higher level, numbers have become mathematical objects that derive their meaning from a network of number relations (cf. Van Hiele, 1973). In this sense we may speak of the creation of a new mathematical reality, which is constituted by mathematical objects in a network of number relations.

The proactive role of the teacher in common-sense mathematics education

If we accept that mathematical growth coincides with constructing new mathematical reality, we may conceive mathematics education as supporting students in constructing new mathematical reality. This fits with Freudenthal's (1973) notion of "mathematics as a human activity". In his view students should be given the opportunity to reinvent mathematics. The challenge then is: How to make students invent what you want them to invent?

In answering this challenge, I want to put the following claims to the fore:

- If you want *students* to invent significant mathematics, you have to offer them support.
- If you want *teachers* to help students to invent significant mathematics, you have to offer them (the teachers) support.

If you want *textbook authors* to help teachers to help students to invent significant mathematics, you have to offer them support

To substantiate these claims I want to start by analyzing the proactive role of the teacher. What can a teacher do to help students invent certain pieces of mathematics? Following Cobb, Yackel & Wood (1992), we may discern the following actions:

- Designing /choosing instructional activities and tools.
- Establishing an "inquiry math" classroom culture.
- Framing topics for discussion and orchestrating whole-class discussions.
- Making connections with the mathematical conventions and practices of the wider community.

In addition, the teacher has to take care of many other things, such as managing the classroom, and organizing group work and individual work.

We may partition this list according to two elements, planning and acting.
Planning:

- Designing/choosing instructional activities and tools.

Acting:

- Establishing an "inquiry math" classroom culture.
- Framing topics for discussion and orchestrating whole-class discussions.
- Making connections with the mathematical conventions and practices of the wider community.

I want to start by looking at the action side, and in doing so I will focus on what instructional practices might be needed to foster the construction of a new mathematical reality by the students. More specifically, I want to focus on establishing an "inquiry math" classroom culture. In doing so, we may hook up with the notion of "didactical contract" of Brousseau (1988), and the idea of social norms and socio-mathematical norms as proposed by Cobb and Yackel (1996).

Establishing an inquiry mathematics culture

The didactical contract refers to an implicit agreement between teacher and students about the rules of the game; "What does it mean to teach and learn mathematics in school?" A very familiar contract is that of traditional "school math". Typical here is the so-called elicitation pattern. How typical this is may be illustrated by conceiving how this pattern would play out in everyday-life situations. Imagine the following.

You are walking around in Copenhagen and a stranger approaches you with the question:

> "Could you please tell me the way to the Tivoli?"

You happen to know this, and you answer:

> "You take the second on the right, then the first on the left, and then you work straight into the Tivoli Gardens."

The stranger answers:

> "OK, the second on the right, then the first on the left, and then straight ahead. Perfect, well done."

And he goes on by asking:

> "Now, could you also tell me the way to Christianshavn?"

You would think this a stranger indeed, but in a mathematics classroom this would be quite a normal pattern. It is quite normal that the teacher asks questions to which he or she knows the answer, and of which the students know that the teacher knows the answer. The students expect the teacher to evaluate their answer, for that is part of the implicit contract.

Let me illustrate this with another example from a study by Elbers (1988). Kindergarten students were asked: What is heavier, red or blue? The interesting thing here is not whether the students thought either blue or red heavier, but that they gave an answer. The explanation here is that these young children are already permeated with the rules of the game. They know what is expected from them: They have to give an answer, any answer. They know from experience that they often do not know what the answer is, nor why this specific question is being asked. They do know, however, that the teaching-learning process comes to a hold if they do not give an answer.

The basis of the traditional school-mathematics didactical contract is that the students have to come to grips with knowledge the teacher already has. The teacher's role is to explain and clarify; the students' role is to try to figure out what the teacher has in mind.

An "inquiry mathematics" didactical contract is completely different. Here the students have to figure out things for themselves, and instead of giving them answers, the teachers may ask them new questions. A key point here is the intellectual autonomy of the students (Kamii, Lewis and Livingstone Jones, 1993). Instead of having the teacher or textbook as the authority, the classroom has to self-reliantly co-construct mathematical knowledge as a learning community. This brings with it the obligations for the students to explain and justify their solutions, to try to understand the explanations and reasoning of the other students, to ask for clarification when needed, and eventually to challenge the ways of thinking with which they do not agree.

This does not mean, however, that the teacher has no authority at all. On the contrary, the teacher is the one who determines what it means to learn mathematics in this classroom. The teacher determines what mathematics is, what mathematical arguments are and so forth. In addition, the teacher guides and supports the process by posing tasks, framing topics for discussion, orchestrating discussions, and when needed, making connections with the mathematical conventions and practices of the wider community.

How to change the didactical contract?

The above implies that teachers and students will have to make a shift, and change the didactical contract from school mathematics to inquiry mathematics. This takes some conscious effort. Research shows that students often resist teachers' attempts to implement a problem solving approach (Desforges and Cockburn, 1987; Stigler and Hiebert, 1999).

The dual perspective of Cobb and Yackel (1996) may help us to understand this resistance.

Cobb and Yackel propose to coordinate a social perspective and a psychological perspective, with the former taking the classroom as the unit of analysis and the latter the individual student. They observe that on the one side the classroom social norms are constituted by the beliefs of the individual students, and on the other side the individual beliefs are shaped by the classroom social norms. Changing the social norms therefore implies changing the individual beliefs, while the students' beliefs about their obligations and those of the teacher are formed by experience. The students are used to being rewarded for reproducing the teachers' reasoning and procedures, and they will keep believing that things will stay that way unless they experience otherwise.

To establish new social norms the teacher has to convince the students that what is valued and what is rewarded has changed. One way to do so is to use concrete instances as opportunities to clarify norms.
Let me illustrate this with an example (taken from Yackel, personal communication).

A second grade classroom discusses a mathematical problem to which the correct answer is eight. The following exchange unfolds.

Mr. K.: "How many?"
Donna: "Eight"
Mr. K.: "How many?"
Donna: "Eh, … seven (?)"

Next Mr. K. moves to other students. Later, as it is established that eight was the right answer, Donna complains.

Donna: "I said eight but you said I was wrong!"

Mr. K.: “What is your name?”
Donna: “Donna”
Mr. K.: “What is your name?”
Donna: “Donna”
Mr. K.: “And if I would ask you again, “What is your name?” would you say anything else but Donna?”

Next the teacher, Mr. K., explains to Donna that if she thinks her answer is correct, she should stick to that answer, whether it is her name, or an answer to a mathematical problem. In this manner, Mr K. works on establishing the social norms that he strives for, by explicating to the students what he expects from them.

Socio-mathematical norms

Next to the social norms, Cobb and Yackel (1996) also discern socio-mathematical norms. These are classroom norms specific to mathematics. Examples of such norms are:

- what counts as a mathematical problem
- what counts as a mathematical solution
- what counts as a different solution
- what counts as a more sophisticated solution.

By establishing these norms, the teacher provides the students with criteria for judging arguments and solutions. This is very important for the intellectual autonomy of students. Thanks to such criteria the students can make their own evaluations and do not have to wait for the final judgment of the teacher. This may clarify an earlier point I made about the authority of the teacher, by establishing the socio-mathematical norms; the teacher defines what mathematics is.

Let me clarify the socio-mathematical norms a little further with some examples. Here we may first look at what counts as a mathematical problem, and in connection with that look at what counts as a mathematical solution. These are key issues in mathematics education since students are expected to take reality into account while at the same time keeping an eye on the mathematics.

Verschaffel, Greer and DeCorte (2000) have done extensive research on the manner in which primary school students take reality into account when solving word problems. One set of problems looks like this:

Jim has 5 planks of 2 meters.
> How many planks of 1 meter can he make?

John has 4 planks of 2½ meters.
> How many planks of 1 meter can he make?

Verschaffel and his colleagues found that many students give the same answer to both problems: 10 planks. In the second case, this answer is not very realistic. We may, however, also wonder about 10 as an answer to the first problem. If we would take into account the width of the cut made by the saw, this answer would not be correct.

Apparently there is an implicit understanding among educators that is not expected. Still it is a subtle issue, how the student is supposed to know?

We cannot give detailed rules that cover every case. Instead the students will have to learn from experience of dealing with such issues in the classrooms what the norms are.

I would like to discuss another example. The task is the following.

> *Mary's friend Ann is staying for diner, now there are 5 cheeseburgers for 6 people (father, mother, Mary, her brothers and Ann).*
> *> How should they share the cheeseburgers?*

Discussions with adults showed me that there are various real-life solutions, which differ from what we are aiming for in mathematics education. Solutions such as:

- Mary should share with Ann, it is her friend.
- Go to the shop and buy one extra.
- Everybody, except Ann's mother gets a cheeseburger, and all give a small piece to her.

In addition to this some would divide the cheeseburgers in a very practical manner:

- First divide three burgers into two, which gives you six halves, then divide the remaining two each into three parts.

A more school-type solution might be:

- Divide all cheeseburgers into six pieces and give everyone five pieces.

The first two solutions are not what you want in a mathematics classroom. You want the other three solutions that are mathematically productive, for they correspond with three different number sentences (figure 2).

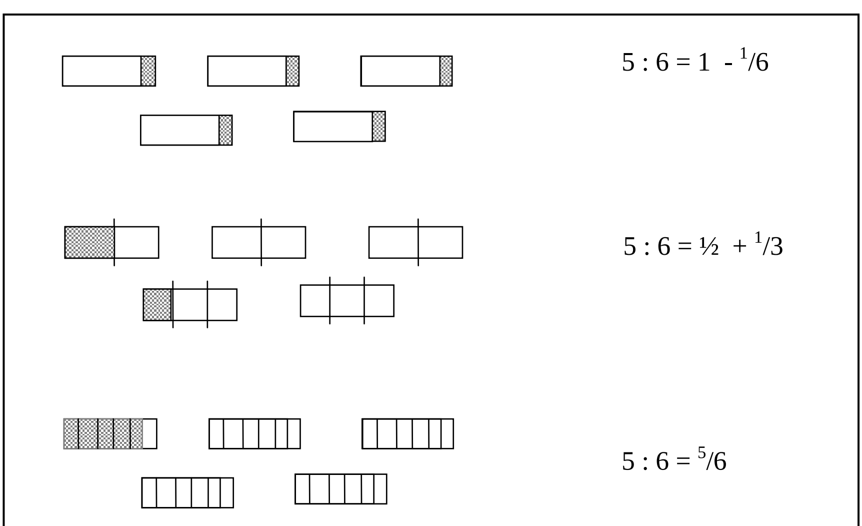

Figure 2. Mathematically productive solutions**.**

We may conclude that socio-mathematical norms about the extent to which reality has to be taken into account have to be developed in interaction. In general, we may argue, all classroom norms are based on experience. Experience of what is valued, and what is rewarded by the teacher.

Designing or choosing instructional activities and tools

Earlier I made a distinction between planning and activity. Above I discussed the didactical contract as an essential element of an instructional practice within which student invention may prosper. In the following, I want to turn to the other side of the medal, planning.

In the type of instruction that is labeled as "transmitting knowledge", planning is rather straightforward since the teacher is the one who transmits the knowledge. In reform mathematics education, however, things are more complicated since it is the students who have to do the inventing.

Teachers can influence their students' inventing activity only in a more indirect manner. To do so, teachers will have to put themselves in the shoes of the students. This asks for a shift from an *observer's point of view* to an *actor's point of view* (Cobb, Yackel and Wood, 1992), where the actor is the student, and the observer the teacher. The challenge for the teacher—and also for us—is to try to see the world through the eyes of the student. How much these worlds may differ may be illustrated by other pictures of Watson's strip about Calvin and Hobbes. Figure 3a shows the world of Calvin and his tiger friend Hobbes seen through his eyes, and figure 3b shows how we see Calvin and his tiger doll.

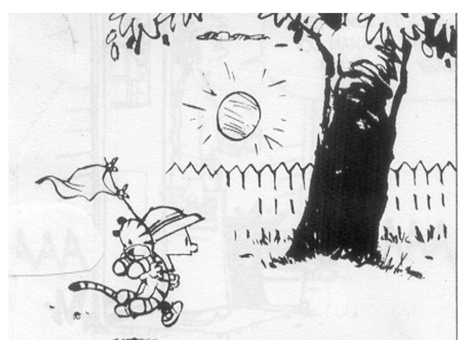

Figure 3a. Actor's point of view **Figure 3b. Observer's point of view**

(Copied from Bill Watson (1996), *It's a magical world*, Kansas City: Andrews and McMeal, page 50 and 82.)

To be able to plan instructional activities that may foster certain student inventions, the teacher has to take an actor's point of view, and to try to anticipate what students might do. In this manner, the teacher can plan instructional activities that may foster the mental activities of the students, and which fit his or her pedagogical agenda. In relation to this, Simon (1995) speaks of a hypothetical learning trajectory. The notion of a hypothetical learning trajectory entails that the teacher has to envision how the thinking and learning, in which the students might engage as they participate in certain instructional activities, relate to the chosen learning goal.

Apart from the aspect of anticipating the mental activities of the students, a key element of the notion of a hypothetical learning trajectory is that the hypothetical character of the learning trajectory is taken seriously. The teacher has to investigate whether the thinking of the students actually evolves as conjectured, and he or she has to revise or adjust the learning trajectory on the basis of his or her findings. In relation to this, Simon (1995) speaks of a mathematical teaching cycle. In a similar manner, Freudenthal (1973) speaks of thought experiments that are followed by instructional experiments in a cyclic process of trial and adjustment.

If we accept this image of the role of the teacher in instruction that aims at helping students to invent some (to them) new mathematics, we may ask ourselves, what type of support should be offered to teachers. Apparently, we will have to aim at developing means of support that teachers can use in construing and revising hypothetical learning trajectories. Such a means of support might consist of a combination of a set of exemplary instructional activities that can be used to teach a certain topic (such as, for instance, fractions, long division, or data analysis), and a rationale, or local instruction theory (Gravemeijer, 1998) that underpins it.

Local instruction theories are the product of design research within which prototypical instructional sequences are developed in a cumulative process of designing and revising instructional activities. These local instruction theories encompass both theories about the process of learning specific topics and theories about the means designed to support that learning. In addition, the local instruction theories also define the required classroom culture.

Mark that, local instruction theories differ significantly from conventional teacher guides. The local instruction theories do not comprise scripted lessons. Instead, local instruction theories will have to offer frameworks of reference for designing hypothetical learning trajectories. For—even with local instruction theories available—teachers will still have to construe their own hypothetical learning trajectories. Each hypothetical learning trajectory will have to be tailored to the actual situation of *this* teacher and *these* students and at *this* moment in time.

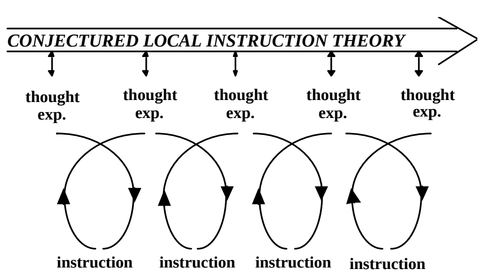

Figure 4. Local instruction theory and thought and instruction experiments.

Local instruction theories are, as already mentioned, developed in design research projects. At the core of each design research project is a classroom teaching experiment within which a prototypical instructional sequence is developed in an iterative process of designing and revising instructional activities. The activity of designing instructional activities is guided by a conjectured local instruction theory, which is developed in advance, and which is refined and adjusted in the process (see figure 4).

Actually, there is a reflexive relation between the local instruction theory and the thought and instruction experiments, as the theory gets revised and adjusted under influence of what is learned in the process. Finally, an improved local instruction theory will be reconstructed in the retrospective phase of the design research project. Then the cycle may start again, the cycle of preliminary design, teaching experiment and retrospective analysis can be repeated several times, in order to improve the (conjectured) local instruction theory.

RME design heuristics

Developmental, or design, research at the Freudenthal Institute, and elsewhere, has resulted in a domain-specific instruction theory for realistic mathematics education (RME), which is grounded in numerous concrete elaborations. We may characterize this theory by three design heuristics: guided reinvention, didactical phenomenology, and emergent modeling.

According to the heuristics of *guided reinvention* a route has to be mapped out along which the students can (re)invent the intended mathematics by themselves—or at least can experience the process as such. Here the researcher/designer may look at the history of mathematics as a source of inspiration (Freudenthal, 1973), or look at informal solution procedures of the students that may be interpreted as anticipating more formal mathematics (Streefland, 1990).

The *didactical phenomenology* design heuristic asks for a didactical phenomenological analysis. The basis is in a phenomenology of mathematics, in relation to which Freudenthal (1983) speaks of mathematical “thought things” (which may be mathematical concepts, procedures or tools) that organize certain phenomena. The task then is to find out how the mathematical thought thing organizes the corresponding phenomenon. Knowing how the thought thing organizes certain phenomena, the designer may create a situation within which these phenomena “beg to be organized” as Freudenthal (1983) puts it.

Assuming that mathematics has emerged as a result of solving practical problems, we may presume that the present-day applications encompass the phenomena which originally had to be organized. Consequently, the researcher/designer is advised to analyze present-day applications in order to find starting points for a reinvention route. Such starting points may, for instance, consist of problem situations that may give rise to situation-specific solution procedures, which in turn can be mathematized to arrive at conventional mathematical procedures.

The design heuristic of *emergent modeling* refers to the notion of a model that may come to the fore as a model of informal mathematical activity and over time may develop into a model for more formal mathematical reasoning.

I will elaborate on this emergent modeling heuristics in greater detail in the following.

Emergent modeling

To clarify what is meant by emergent modeling, it may be fruitful to distinguish three types of models, *didactical models*, *mathematical models* and *emergent models*. What I call didactical models, here, may be thought of as concretized expert knowledge. A rather common strategy in mathematics education is to try to concretize abstract, formal, mathematical concepts in concrete manipulatives or visual schemas. The idea then is that these abstract concepts are made easier to grasp by making them concrete. The problem with this approach, however, is that only experts can see the intended formal mathematical concepts in the concrete materials. Only those who already have the concept see it.

A completely different type of model is the mathematical model. In applied mathematics one will construe a mathematical model of a problem situation in order to make predictions, or solve some problem. This does not ask for the invention or appropriation of new mathematics; the problem solver uses the mathematics he or she already has available to construct the model. Another characteristic is that the model and the situation modeled are treated as two distinct entities. In this the mathematical model differs from the emergent model. With emergent modeling, there is not such a clear distinction.

In emergent modeling "the model" and the conception of that is being modeled co-evolve, and initially, there is no differentiation between the model and that what is being modeled. I have put model between quotation marks, as there is not *one* model in a process of emergent modeling but a series of symbolizations or sub-models that together constitute "the model". From the perspective of the researcher/designer, it is this overarching model that develops from a *model-of* informal mathematical activity into *model*-for more formal mathematical reasoning. For the students, the emphasis is on modeling as a process, as an organizing activity.

We may describe the process that is aimed for with the help of the various levels of student activity that may be discerned (figure 5). The starting point is in activity in the task setting (*situational level*), in which interpretations and solutions depend on understanding of how to act in that setting. These will often be out-of-school settings. In reasoning about such a situation in the context of school tasks, a situation-specific model may be needed, which can be used to model such a situation (*referential level*). Here we may speak of referential activity, in that the model derives its meaning for the students from its reference to activity in the task setting, thus the model functions as a *model of* that activity.

Gradually, and with help of the teacher, the attention may shift towards the mathematical relations involved (*general level*). As a consequence, the model starts to derive its meaning from those mathematical relations, and in this manner, the model starts to become a *model-for* more formal mathematical reasoning. Finally the students may reach the level of more formal mathematical activity (*formal level*), when more formal mathematical reasoning is no longer dependent on the support of a model.

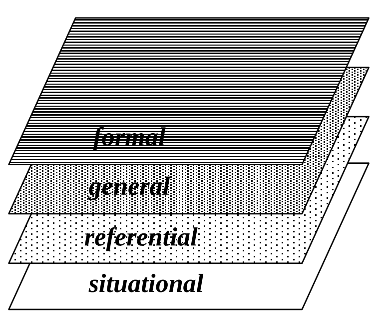

Figure 5. Levels of mathematical activity.

Here, "more formal" is meant to refer to the constitution of new mathematical reality. More formal mathematical reasoning is more formal in that it relates to a new mathematical reality—or to put it differently, to a framework of mathematical relations that is new to the students.

From the perspective of the researcher/designer there is an over arching model, which at first is constituted as a context-specific *model of* acting in a situation, then changes in character and becomes an entity of its own right for the students. Having become an entity of its own right for the students deriving its meaning from the newly formed mathematical framework, the model can start to function as a *model for* more formal mathematical reasoning.

The emergent modeling design heuristics asks from the researcher/designer to conceive of a model that can play this role, think through the series of symbolizations or tools that will have constitute the model, and explicate the new mathematical reality, or the framework of mathematical relations and the mathematical objects that have to be construed by the students. The latter explication is important, as it has to inform the teachers about what mathematical issues have to be framed as topics for discussion in the classroom.

Exemplary instructional sequence

I will illustrate the aforementioned design heuristics with an exemplary instructional sequence (see also Gravemeijer, 2004). The objective of this instructional sequence is to help students to develop flexible strategies for addition and subtraction up to 100.

According to the guided reinvention heuristics, we may ask ourselves how students might invent the strategies we are aiming for. The answer may lay in the realization that these strategies are based on a flexible use of a network of number relations, within which multiples of ten will be significant reference points. Thus the task will be to help students develop this network of number relations. To do so, we will want to start with the informal solution procedures of students.

Research shows that there are two main strategies, splitting tens and ones, and counting in jumps (Beishuizen, 1993). Following the first strategy one would solve the addition 44 + 37 = … by first adding the "tens", 40 + 30 = 70, then adding the "ones", 4 + 7 = 11, and then putting the results together, 70 + 11 = 81.

Following the second strategy one would keep the first number intact and add the addend in steps, e.g., via 44 + 30 = 74; 74 + 7 = 81, or: via 44 + 6 = 50; 50 + 10 = 60; 60 + 10 = 70; 70 + 10 = 80; 80 + 1 = 81.

Research also shows that the strategy of splitting tens and ones is prone to errors in the case of subtraction (Brown and Van Lehn, 1982). We may therefore opt for helping the students in inventing the strategy of counting in jumps.

The didactical phenomenology design heuristics points us to the applications. On the basis of his phenomenological analysis, Freudenthal (1973) distinguishes various number concepts:

- label number
- counting number
- quantity number
- measuring number
- arithmetical number.

He observes that we use the counting number to solve problems about quantities. He further notes the practical relevance of the measurement number, and the fact that measurement integrates both the counting and the quantity number. He concludes that one may use measurement in combination with the number line as a situation where the students, too, can make the connection between the counting and the quantity number.

An instructional sequence that was developed along these lines is an instructional sequence on linear measurement and on addition and subtraction up to 100, which was developed in a teaching experiment at Vanderbilt University (Stephan, Bowers, Cobb, and Gravemeijer, 2003; Cobb, Stephan, McClain and Gravemeijer, 2001; Gravemeijer, McClain and Stephan, 1998). In a nutshell, the sequence consists of the following steps:

- measuring by iterating some measurement unit
- measuring with tens and ones (coordinating. 10s and 1s)
- modeling the activity of iterating with a ruler
- reasoning about measures (while taking a measure as a given entity):
    - incrementing, decrementing and comparing lengths
    - developing arithmetical solution methods
- symbolizing the arithmetical solution methods with arcs on a schematized ruler (empty number line)
- using arcs on an empty number line as a way of scaffolding, and as a way of communicating solution methods for all sorts of addition and subtraction problems.

The activity of symbolizing arithmetical solution methods with arcs on an empty number line may be illustrated with the task presented in figure 6.

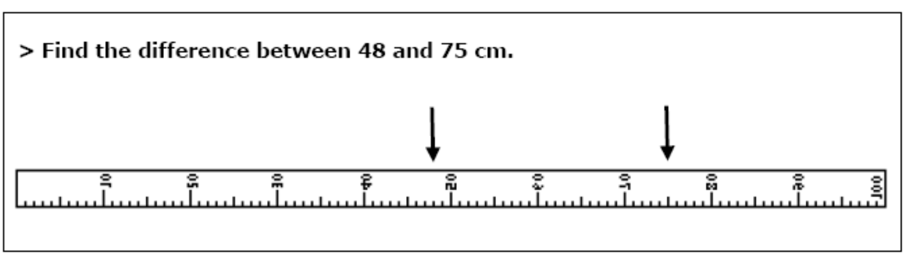

Figure 6. Reasoning about measures.

Here one may reason,

48+2=50,
50+10=60,
60+10=70,
70+5=75.

Thus the difference is, 2+10+10+5=25 cm.

This kind of reasoning can be depicted nicely with arcs on an empty number line (see figure 7).

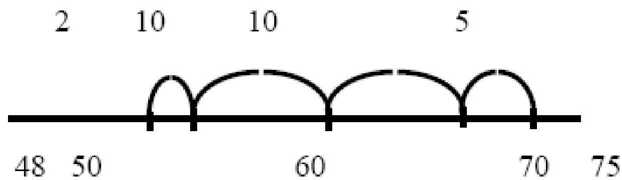

Figure 7. Arcs on the empty number line.

This is called an *empty* number line (Treffers and De Moor, 1990), because the idea is that the student starts with an empty line, and adds numbers as the reasoning unfolds. So, the student starts with adding a hash mark and writing 48 underneath it, next he or she draws the arc for the addition of 2, and writes 50 under the new mark, and so forth. In this manner the empty number line offers a perfect tool for expressing the student's personal reasoning.

Figure 7 depicts the calculation of the missing addend, 48 + … = 75. Looking at the numbers, one might have come up with a neater solution, using that 75 equals 50 + 25. Here the number line might be used to explain that 75 – 48 is two more than 75 – 50.

In reflection on this sequence, we may observe three interrelated processes. Firstly, there is the model shift: from *model-of* to *model-for*. Here the overarching model is the "ruler". Initially the ruler comes to the fore as a model of iterating measurement units of different ranks (tens and ones), and finally, the schematized ruler, the empty number line, functions as a model for mathematical reasoning. Secondly, there is the process of constructing some new mathematical reality by the students. This new mathematical reality is constituted by numbers as mathematical objects in the framework of number relations. (The number 37 for example derives its meaning from relations such as 37 = 30 + 7 = 3x10 +7 = 20 + 17 = 40 – 3, and so forth.) Thirdly, there is the process of symbolizing, which encompasses a series of symbolizations/tools, such as: the basic measurement unit, the bigger measurement unit of 10 basic units, the measurement strip (paper ruler), and the empty number line. Ideally, the students consciously experience the symbolizing process, which can be furthered by problematizing every symbolizing step. In this manner, the students may experience the emergence of more sophisticated tools as a reinvention process.

A key issue here is that the use of a new tool is always grounded in imagery of earlier activities. This distinguishes emergent modeling from the use of didactical models even though we may compare the role of emergent models as models-for-more-formal- mathematical-reasoning with the role of didactical models. In emergent modeling, the model-for is grounded in the preceding learning process of the student.

It is this history that gives meaning to the model, and it is exactly this history that is missing in the conventional use of didactical models.

The difference with mathematical modeling is based in a difference of objectives. Emergent modeling is meant to help students develop the mathematics they may need to apply later, so the focus is on a learning process. With mathematical modeling the objective is to solve a given problem. Or to put it differently, emergent modeling may help the students develop the mathematical toolbox that they will need for mathematical modeling. In addition, being involved in emergent modeling, the students may develop a problem-solving attitude that may they need with mathematical modeling.

Conclusion: Creating opportunities for students to reinvent mathematics
I want to conclude by returning to the claims I made in the beginning of this chapter:

- If you want *students* to invent significant mathematics, you have to offer them support
- If you want *teachers* to help students to invent significant mathematics, you have to offer them support
- If you want *textbook authors* to help teachers to help students to invent significant mathematics, you have to offer them (the teachers) support

I have shown that if you want students to invent some new mathematics, several things have to be in place. I especially focused on the classroom culture and the instructional activities. In relation to this, we may conclude that if we want to help teachers, we will have to make them aware of the didactical contract and of the connected social norms and socio-mathematical norms. Moreover, we will have to offer support in clarifying how the teacher can establish the social norms that correspond to an inquiry mathematics classroom culture.

As far as the instructional activities are concerned, I emphasized the importance of planning: If you want to capitalize on the input of the students, you have to plan ahead. Planning is needed to ensure valuable (useful) student input. In this respect, I pointed to the notion of a hypothetical learning trajectory. To plan instruction that builds on the ideas and input of the students, the teacher has to envision the mental activities that the students might engage in when they would participate in the instructional activities considered. Here, the teachers will have to take into account their own goals and preferences, the characteristics of the students, and their learning history. However, it seems unreasonable to ask from teachers to design such hypothetical learning trajectories from scratch. We may help teachers by developing local instruction theories—together with exemplary instructtional activities—which the teachers can use as a framework of reference for designing hypothetical learning trajectories.

I argued that design research is tailored to developing this kind of local instruction theories, and I showed that RME offers design heuristics that can be used to develop local instruction theories that fit the idea of realistic mathematics education. In this respect, RME theory offers an alternative for the classical design theories, and with those heuristics, and with the local instruction theories available, RME theory offers means of support for instructional designers and textbook authors.

As a final remark, I want to address the concern of some that the notion of learning trajectories and local instruction theories is too rigid, too linear. I want to counter, however, that we cannot expect all students to follow the same learning trajectory in unison. There will be variation in both solution procedures and student understanding. More importantly, variation in solution procedures and student understanding is indispensable, the whole-enterprise is directed at designing and choosing those instructional activities that will generate variation. For the contrasts in student thinking form the leverage points for whole class discussion, and as such the basis for progress. This is exactly why planning is so important, because we want to ensure this variety. If all the students would come up with the same kind of reasoning, there would be nothing for the teacher to build on. On the other hand, I would also argue that the variation should not be too large. That would be impossible to handle for the teacher. To conclude, this is why I think that it is so important to help teachers in designing hypothetical learning trajectories.

References

Beishuizen, M. (1993). Mental Strateges and materials or models for addition and subtraction up to 100 in Dutch second grades. *Journal for Research in Mathematics Education,* **24** (4), 294-323.

Brousseau, G. (1988). Le contrat didactique: le milieu. *Recherche en didactique des mathématiques* **9**(3), 33 – 115.

Brown, J.S. & Van Lehn, K. (1982). Towards a generative theory of 'bugs'. In T.P. Carpenter, J.M. Moser, & T.A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective.* Hillsdale, NJ: Lawrence Erlbaum Associates, 117—135.

Cobb, P., Yackel, E. & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education,* **23 (**1), 2—33.

Cobb, P. & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist,* **31**, 175-190.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. P. E. (2001). Participating in Classroom Mathematical Practices. *Journal of the Learning Sciences,* **10**(1&2), 113-163.

Davis, Ph. & Hersh, R. (1986). *Descartes' Dream.* Boston: Houghton Mifflin Company.

Desforges, Ch. and A. Cockburn (1987), *Understanding the Mathematics Teacher, A Study of Practice in First School,* London: The Falmer Press.

Elbers, E. (1988), *Social context and the child's construction of knowledge,* Utrecht (doctoral dissertation).

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht: Reidel.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrecht: Reidel.

Freudenthal, H. (1991). *Revisiting Mathematics Education*. Dordrecht: Kluwer Academic Publishers.

Gravemeijer, K., McClain, K., & Stephan, M. (1998). Suporting students' construction of increasingly sophisticated ways of reasoning through problem solving. In: A. Olivier & K. Newstead. *Proceedings of the 22nd conference of the International Group for the Psychology of Mathematics Education, Volume 1*, 1-194 - 1-209.

Gravemeijer, K. (1998). Developmental Research as a Research Method, In: J. Kilpatrick and A. Sierpinska (Eds.) *Mathematics Education as a Research Domain: A Search for Identity (An ICMI Study).* Dordrecht: Kluwer Academic Publishers, book 2, 277-295.

Gravemeijer, K. (2004). Learning Trajectories and Local Instruction Theories as Means of Support for Teachers in Reform Mathematics Education. *Mathematical Thinking and Learning,* **6**(2). 105-128.

Kamii, C. en Lewis, B.A. & Livingstone Jones, S. (1993). Primary Arithmetic: Children Inventing Their Own Procedures. *Arithmetic Teacher,* **41**(4), Columbia, NY: Teacher College Press, 200-203.

Simon, M.A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education,* **26**, 114-145.

I C M E
1 0
2 0 0 4

RL

Regular Lecture

Stephan, M., Bowers, J., Cobb, P., & Gravemeijer, K. (2004). *Supporting students' development of measuring conceptions: Analyzing students' learning in social context.* Journal for Research of Matematics Education Monograph 12.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap*. New York: Free Press.Streefland, L. (1990). *Fractions in Realistic Mathematics Education, a Paradigm of Developmental Research.* Dordrecht: Kluwer Academic Publishers.

Treffers, A. en De Moor, E. (1990). *Proeve van een nationaal programma voor het reken-wiskundeonderwijs op de basisschool. Deel II. Basisvaardigheden en cijferen.* Tilburg: Zwijsen.

Van Hiele, P.M. (1973). *Begrip en Inzicht*. Purmerend: Muusses.

Verschaffel, L., Greer, B., & de Corte (2000). *Making sense of word problems*. Lisse: Swets & Zeitlinger.