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Dynamic structural learning: From theory to practice

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Introduction

A question frequently addressed by cognitive researchers and neuroscientists is “How does learning take place?” A large gamut of models explaining mind changes through learning has been proposed since the domain of theories of learning emerged in the first half of the twentieth century. While the proposed answers inevitably proved limited, modeling only partially the complexity of the problem, a more feasible question, addressed by both educators and researchers can be “How could learning be optimized?” This question moves the issue from the general domain to the domain of specific approaches of cognition. Thus, in mathematics, many educational research programs have developed since the 70’s (e.g.: Dienes, 1963; Freudenthal, 1973; Schoenfeld, 1985). However, the research has usually been done outside the classroom, with a small number of students, and focused on a well specified topic (for example, elements of topology, or elements of algebra, or understanding motion). When the research happened in class, it usually focused on micro-ethnographical studies of teacher-student interactions (e.g. Brousseau, 1980; Cobb and Bauersfeld, 1995). Although it is a problem that confront the teaching of each generation of students, educational researchers very rarely address the question: “How can learning of an entire domain of knowledge be optimized?”. Experiments of great breadth dealing with the learning of a school subject are difficult to track because of the multitude of the variables interfering in the teaching-learning process in the classroom; however, long-term experiments are the only way to address the constraints interfering in a real school context. This essay focuses on one such a long-term experiment, the dynamic structural learning experiment, and provides some insights into its application in the primary grades 1-4.

In a synthetic description, the *dynamic structural learning* (DSL) of a domain is seen as a bipolar cognitive construction: on the one hand, the domain to be taught is organically integrated in a structured system that is focused on developing a specific way of thinking; on the other hand, learning each subset of the system implies an *active reconstruction* of its semantics (in the sense of the constructivist definition of learning). This type of learning aims at building dynamic thinking structures that are capable of generating in learners, under appropriate conditions, an expert behavior (Singer, 2001a).

Children in primary grades represent a good target population on which to apply the theory because their mental structures are not yet affected by other learning procedures. This allows us to evaluate learning progress more accurately. In addition, students’ mental flexibility at young ages allows for learning errors to be more easily corrected. Conversely, rigid structures already in place generate in older students mental engravings that are difficult to remove. Given its cognitive focus, I hypothesize that the systematic use of DSL from primary up to secondary school will remarkably increase students’ creative potential.

A formal definition of dynamic structural learning cannot sufficiently cover its complexity. For this reason, the paper will present in some detail the input components of the theoretical model, along with some examples. The DSL involves the training of three conceptually independent components: mental operations, information, and layers of abstracting, which are seen as generative vectors of the domain-specific mental space.A visual representation of this triplet as a cuboid will emphasize the different structures of the components (see Figure 2 in the next section). The cuboid representation will also highlight sections (i.e. planar cuts) that uncover multiple uses of the model and its functionality in developing teaching and learning materials.

While the first level of description refers to the input components, the next level of description focuses on the cognitive outcomes of applying the DSL curriculum. The DSL aims at developing dynamic mental structures. In order to clarify the latter concept, a classification based on a list of parameters allows for a differentiation among types of structures based on their mobility. To show how such a model can work in practice, a snapshot of the experimental teaching is then briefly presented, together with some results of a longitudinal study that followed the DSL specific methodology.

The input of learning. The triplet: Mental operations, information, layers of abstracting

Recent findings in cognitive science and neuroscience have shown that the sense of numerosity is present in infants before and independent of language, giving new arguments for the possibilities of structuring innate predispositions. Thus, for example:

5 months old infants seem to be able to compare two sets of up to three objects (e.g.: Feigenson, Carey and Hauser, 2002; Koechlin, Dehaene and Mehler, 1997; Starkey, 1992; Wynn, 1992, 1998); the sense of approximating numbers found in adults appears to be present and functional in 5 to 10-month old infants (e.g.: Dehaene, 1997; Karmiloff-Smith, 1992; Spelke, 2002). Infants are able to discriminate between different numerosities for both spatial arrays and temporal sequences in a variety of sensory modalities. A “numerical precocity”, beyond the curriculum requirements, is also identifiable in toddlers and youngsters.

Research on young children’s capabilities of “deciphering the world” invites us to reconsider early learning in school. It shows that the cognitive system is not a blank slate; it already has some content. Here a question becomes legitimate: If there is such a natural endowment, why do so many students have difficulties in learning mathematics? DSL offers answers to this question, including things such as why, what, how, and how much should be taught. DSL aims to develop in the learner competences similar to the ones possessed by the “expert”. It is important to stress that it is not about teaching the “novice” everything the “expert” knows; we are not speaking here about the informational level, but the formational one: the learner will be able to react when facing a problem fitted to his/her mental age and knowledge as the expert addresses a problem in his/her field. This is possible because, while specialized information cannot be “lowered down” to young ages beyond certain limits, specific behaviors can be lowered from the expert level to the (young) novice, providing adequate conceptual and methodological inputs are present.

From the DSL perspective, learning integrates three conceptually independent components: *mental operations* (that are in a permanent process of development and transformation starting from a few operational nuclei), *information* (that is there already, or is to be acquired), and *layers of abstracting* (that are incorporated in both information and operations). The DSL design proceeds from an analysis and classification of these three components.

Mental operations as an input component of learning

The human mind is able to process a multitude of operations. Part of trying to optimize this process is to determine which operations are foundational, that is, which operations can be the basis for the others. I have synthesized, based on literature reviews and experimental use of some learning materials, seven independent clusters of mental operations significant from the point of view of the DSL process (Singer, 2001a): *Associating, Comparing, Algebraic operations, Logical operations, Topological operations, Iterating*, and *Generating*. These constitute what is called *the dynamic infrastructure of mind*. A short description of these operational categories is presented below.

The following operations belong to the category of ***Associating***: *recognizing* (associating an entity with a mental image of it); *naming* (associating an entity to its name); *reproducing* (associating an entity with a shape that preserves its properties, be that shape graphical, physical, or verbal); *representing* (associating an entity with a symbolic image of it); *interpreting* (associating an entity with a descriptive version of it); *classifying* (associating an entity with the class it belongs to). In addition to the correspondences described above, the human mind is able to make many other associations, even if they are not designated in the daily language. Their training can be done stressing the idea of one-to-one association. The capacity of building one-to-one correspondences evolves from its primitive form of *matching* objects one-to-one to associating various representations through isomorphisms.

The category of ***Comparing*** refers to the action by which one relates two entities in order to characterize the similarities and the differences between them. Some basic mental operations belonging to this category are: *estimating* (the action by which one relates the size of a dimension of an entity or the result of a procedure with one or more measures “predicted” by qualitative assessments; *selecting-discriminating* (the action by which one separates an element with certain features out of an unstructured set); *checking* (the action by which one relates an object to an implicit or explicit pre-existent model of it); as well as *numerical comparison*.

The category of ***Algebraic operations*** includes actions that refer to combining quantities in a specific well-defined way in order to get a result that is analyzed from a discrete quantitative perspective. In school, one studies the binary operations (defined on a Cartesian product with two factors), such as addition and multiplication. The *pre-arithmetical* or *proto-quantitative operations* refer mainly to a list of operations that, quantitatively expressed, lead to addition, subtraction, etc., such as: grouping, taking away, magnifying, reducing, adding, combining, etc.

The category of ***Logical operations*** extends the current mathematical meaning of logical reasoning to the general human capacity to formulate logical inferences. These inferences may be inductive, deductive, or analogical.

The category of ***Topological operations*** has a pervasive presence in the development in the first years of life. The topological operations allow for identifying boundaries, relating them with discrete components, perceiving objects globally, passing the frontier between discrete and continuous, and developing an intuition of infinity. They account for the convergence of thinking and for global perception. While algebraic and logical operations are dealing with finite and discrete quantities, *topological operations* are addressing indefinite and continuous properties.

Iterating is described as perceiving regularities in variability; the operations in this category allow for *imitation* in young ages, but further for *identifying and developing patterns*. The operational category of *iterating* accounts for the recursive property of mind.

Generating is described as an operational category the elements of which create new entities, previously unknown, starting from entities already known. This category ascertains a kind of readiness to start. This operational category helps the mind to build the leap to learning; it creates *the innate dimension of the motivation to learn.*

The difficulty of describing these categories of operations arises mostly from the fact that some of them do not come with linguistic labels. While research will concentrate on identifying, analyzing and characterizing these operations, a specific terminology will facilitate their description.

The operational categories described above have some *targets* beyond learning a discipline. Thus, *associating* points to building equivalent metaphors. As Lakoff (1987) argues, we share a fixed, conventional system of conceptual metaphors – a system of thousands of "metaphorical mappings", each facilitating our understanding of one domain of experience in terms of another, typically more concrete, domain. In the same vein, *comparing* allows conceptual human beings to assess various metaphors for the same entity and to make transfers from one entity to another, taking various perspectives. Here also the part-whole relationship involved by metonymies is emphasized. Lakoff and Núñez (2000) argue that mathematics extends beyond its “theoretical cover” through the use of metaphors and the demands for some critical elements in a mathematical theory (such as closure, or intersection at infinity) express artifacts of the human mind's ability to metaphorically relate fundamentally different concepts. From this perspective, *associating* and *comparing* target at internalizing individual contextual experience, and address the dynamics of mental acquisitions while experiencing the environment. Further, *algebraic operations* assure the processing of discrete quantities and emphasize a digital approach, while *topological operations* allow us to operate with continuity and emphasize an analogical approach. These two categories allow us to conceptually perceive, assess and combine both the quantum behavior and the wave behavior of matter. These two behaviors are parts of our physical and mental worlds, and in this context, the algebraic and topological operations place a stress on the duality of matter and processes. *Logical operations* allow for the development of a coherent language; conversely, language has a “scaffolding” role by stimulating the development of thinking. This leads to the building of meta-systems of thought in which the logical operations play the role of connection-agents. The *iterating* and *generating* categories conduct, in conjunction and separately, to the creation of the (so-called) emergence states in complex systems (e.g. Bar-Yam, 1997). These last two categories, which focus on developing recursive processes and, respectively, on intrinsic motivation, are the “motors” of learning.

The aims listed above make each cluster necessary in the cognitive system and assure that this minimal list of categories is sufficient to describe the dynamic infrastructure of mind. The table in Figure 1 synthesizes the architecture of the basic mental operations within the proposed model.

Categories of operations	*Associating*	*Comparing*	*Algebraic operations*	*Logical operations*	*Topological operations*	*Iterating*	*Generating*
Some basic elements	Recognizing Naming Reproducing Representing Interpreting Classifying Isomorphic transformations	Estimating Selecting-discriminating Checking Numerical comparison	Pre-arithmetical operations Operations with sets Arithmetical operations Operations with variables	Using logical operators Using quantifiers	Identifying boundaries Identifying limits Identifying convergences	Mimicking Identifying patterns Developing recurrences	Grasping Guessing Conditioned generating
Targets	Building equivalent metaphors	Building cross metaphor systems and metonymies	Operating with discrete quantities Digital approach	Constructing meta-systems intermediated by language	Operating with continuity Analogical approach	Developing recursive processes	Developing intrinsic motivation
Basic connections and symmetries	*Contextual experience*		*Duality of matter/processes*			*Emergence*	

Table 1. The architecture of the basic mental operations

Information as an input component of learning

Information in this article means conceptual and procedural knowledge prescribed in the written curriculum to be learned in school. In any structured field of knowledge, there are different *degrees of information complexity*. Three dimensions are used to characterize information complexity within the DSL process: *nature*, *structure*, and *procedure*. Figure 1 highlights passages within specific scales for each dimension (Singer, 2003).

For the *nature* of information, the criterion is the *level of generalization*: information varies from particular to general. In the DSL process, a plethora of shifts from particular to general and vice versa are practiced.

The *structure* of information is described by the *connections* among epistemological entities (concepts, notions, theories, etc.) which organize information on various clusters, from an unstructured one (with random or no connections) to a structured (systemic) one, where relations if hierarchy, coordination, subordination, causality, etc. are established. Our research showed that passing through as many organizational clusters as possible is decisive for the flexibility of the acquired skills.

Concerning the *procedure*, the criterion is the *kinematics of connections*, which organizes information in categories from canonical to creative procedural actions.

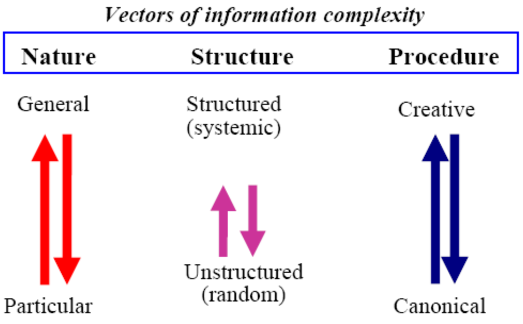


Figure 1. Information degrees of complexity and transfer levels within the DSL model

The layers of abstracting as an input component of learning

Abstracting is a developmental process that supposes a gradual progression from operating with concrete objects to operating with symbols and systems of symbols. This progression also follows a pathway from sensorimotor activity to higher cognitive functions of mind. Complexity and abstraction as attributes of knowledge are in dichotomic opposition: complexity supposes interactions of many underlying factors and emphasis on details, while abstraction supposes simplification and neglecting variables in order to emphasize the essence of a phenomenon or process (Singer, 2007b). For example, the concepts of 3D geometry are more complex whilst the geometry in one dimension is more abstract.

Within the DSL model, the layers of abstracting are both stages in internalizing information and stages in moving from concrete physical representation to abstract representation. Although the passage from one layer to another is usually not explicit, in a working classification, the next layers of abstracting proved helpful in developing teaching-learning materials and strategies:

concrete – at the level of using objects,
unconventional symbolic – at the level of using nonstandard physical or iconic representations,
symbolic – at the level of using standard physical or standard iconic representations,
verbal – at the level of using the linguistic code,
mental – at the level of using internalized representations.

According to the entity they are applied to, it is possible to consider more or less refined scales of abstracting inside the layers.

The cuboid of the basic learning units

The triplet of mental operations, information, and layers of abstracting can be viewed as a three-dimensional structure that develops in three directions. The organization of this structure facilitates developing a curriculum compatible with the DSL focus. For this reason, each element in the structure will be called a *basic learning unit*. Therefore, a basic learning unit is determined by the three criteria listed above, which can be seen as generative vectors: the basic mental operations, information, and the layers of abstracting. These three components have different functions, but also different internal structures. The set of the basic mental operations has a punctiform structure in the sense that each operation is independent, there is no relation, hierarchical or of another type that can be established between the operational categories and between the operations inside the categories. Information has a discrete structure, in the mathematical sense of this word, that is we accept the hypothesis that between two “neighboring” chunks of information there is a certain distance. In other words, when going deeply into the domain, other information may appear and still, a distance remains. The layers of abstracting have an almost continuous structure, that is the “distance” between a certain layer and another one can be infinitely refined, so that the passing becomes practically imperceptible.

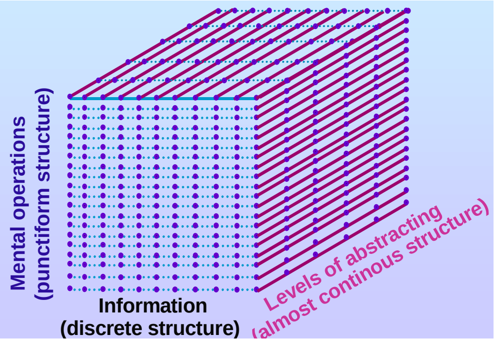


Figure 2: The cuboid of the basic learning units

In order to make the description of this 3D structure more accessible, it is schematically represented by the *cuboid of the basic learning units* (Figure 2). Each “point” (cuboid) in this space, described by the three co-ordinates, represents a basic learning unit. The *cuboid of the basic learning units* is a productive tool for curriculum development, a curriculum compa-tible with a DSL focus. How can this representation be used as a tool? Some examples are given below.

Sections in the cuboid of the basic learning units

Various sections that have specific meanings can be identified in the cuboid.

(a) We can section the cuboid with a horizontal plane parallel to the basis at a “height” corresponding, for example, to the arithmetical operation of addition. Then, we have to focus a “lens” on information, to “select” information according to a specific grade – grade 1, for example - and to track the layers of abstracting. The result is a scale of transition from concrete to abstract regarding learning addition in early ages. The following “12 steps pattern” is emphasized in this way. A more detailed presentation of these steps can be found in Singer (2002.):

1. The first moment is “drama”. Simple actions can be dramatized in the classroom, such as:

    a. *Dynamic animation*. For instance: 3 girls are drawing on the blackboard; 2 boys come and start to draw. (They act the whole scene.) The teacher asks: “How many children are drawing now?” The students in the classroom count: 1, 2, 3, 4, 5.

    b. *Dynamic ‘inanimation’***.** These can take the form of role-playing with objects, for example:

        Dan: “I have two pencils.”

        Ann: “I give you four more. How many pencils have you got now?” (The students play the actions.)

2. *Pretend*. This is preparing the background for thinking in an imaginative plane. (The students mimic contexts in which they put objects together and count.)

The pattern is then developed through gradual variation of representations given by suggestive images:

3. *Explicit active dynamic union* ("How many in total?" based on drawings representing living beings that are moving in a visible way).

4. *Explicit passive dynamic union* ("How many in total?" based on drawings representing inanimate objects that are moved by others in a visible way).

5. *Explicit static union* ("How many in total?" based on drawings representing objects that are linked together in a visible way – like marbles in a necklace).

6. *Implicit static union* (("How many in total?" based on conventional representations used to visualize union, as Venn-Euler diagrams).

7**.** *Active dynamic union in which one of the terms is abstract* (It is similar to (3) but to solve this task category, the child must begin counting not from 1, but from one of the terms).

8. *Passive dynamic union in which one of the terms is abstract (*This is a more abstract version of (4), as (7) is for (3)).

9**.** *Iconic representation* (Additions processed on the number line)

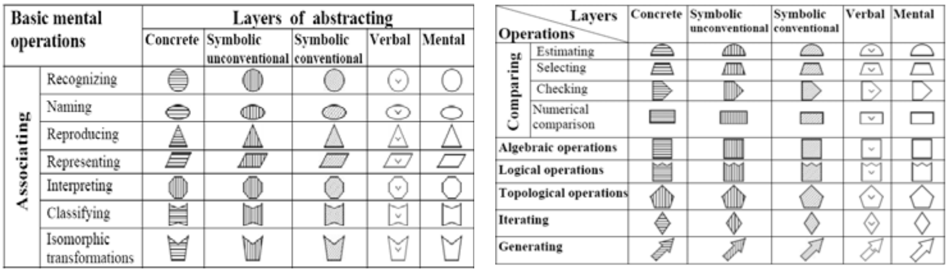
10. *Horizontal symbolic writing of the numbers involved in addition*

11. *Vertical symbolic writing of the numbers involved in addition*

12**.** *Mental computing with no support.*

In the sequence above, the material support of computing acquires a gradually more simplified representation and then disappears. The process of abstracting is following two interfering pathways: one from external environment to internal representation and another, from concrete objects to standard symbolic representations, passing through various unconventional symbolic representations. These pathways are also interfering with language, which plays a Vygotskian scaffolding function, from articulating in the external language (speaking loudly) to "thinking in the mind". The graduation and refinement of these scales are individual functions; to build a norm of them – in a school context we are working with normal distributions – it is necessary to develop a training that emphasizes passages from one stage to another, on different scales, and to review and resume the process periodically.

(b) A vertical section in the cuboid, for a specific bit of information, is visualized in the two tables below, in Figure 3.



Basic mental operations		Layers of abstracting				
		Concrete	Symbolic unconventional	Symbolic conventional	Verbal	Mental
Associating	Recognizing	○	○	○	○	○
	Naming	⬭	⬭	⬭	⬭	⬭
	Reproducing	△	△	△	△	△
	Representing	▱	▱	▱	▱	▱
	Interpreting	⯃	⯃	⯃	⯃	⯃
	Classifying	⧖	⧖	⧖	⧖	⧖
	Isomorphic transformations	▽	▽	▽	▽	▽

Operations \ Layers		Concrete	Symbolic unconventional	Symbolic conventional	Verbal	Mental
Comparing	Estimating	⌓	⌓	⌓	⌓	⌓
	Selecting	⏢	⏢	⏢	⏢	⏢
	Checking	▷	▷	▷	▷	▷
	Numerical comparison	▭	▭	▭	▭	▭
Algebraic operations		□	□	□	□	□
Logical operations		⊓	⊓	⊓	⊓	⊓
Topological operations		⬠	⬠	⬠	⬠	⬠
Iterating		◇	◇	◇	◇	◇
Generating		⇗	⇗	⇗	⇗	⇗

Figure 3. Correlation matrixes between the basic mental operations and the layers of abstracting

The tables in Figure 3 are *generative matrixes*. Replacing the geometrical symbols by specific items, these tables can generate learning activities for various topics to be addressed to students on various levels of mathematical ability.

(c) Sections that are not parallel to the faces of the cuboid are identified and structured through the *transfer operations*. How might the elements of such a non-parallel section look? In order to assure a broader coverage with different examples, I picked up some working tasks referring to various topics, grades and levels. Because DSL focuses on developing creativity, the transfer operations mostly emphasized are based on *generating*. A few examples follow. A more developed list of examples can be found in Singer (2001a), or in the experimental teaching materials.

- *Generating through representation* ("Represent a quarter in different ways". Or "Draw a robot using only squares and rectangles".)
- *Generating through interpretation* ("The number 7 is given. Consider it as being, successively, a sum, a product, the solution of an equation, etc. Devise problems for each case.")
- *Generating through classification* refers to, on the one hand, finding as many classes as possible for a given object, and, on the other hand, finding as many representatives as possible in a given class (for instance, to enumerate as many different objects as possible that might have the same use): ("Draw geometrical figures in which a triangle should be formed by: the diagonals of a polygon; the two sides and one diagonal of a polygon; the intersection of two polygons; the face of a pyramid; the base of a prism, etc."; "Find ten natural numbers that are multiples of 2 and 3 and not of 12.").
- *Generating through estimation* ("Give examples of exercises the results of which are closer to 500 than to 600." "Give examples of objects the height of which is about 2m.")

Other types of tasks may refer to: *generating through analogy; generating through substitution; generating through comparison; generating through counting; generating through selection-discrimination; generating through arranging; generating through increasing and decreasing; generating through combining;*

generating through logical derivation; generating through changing (varying) the hypothesis; generating through changing (varying) the conclusion.

The learning process: Developing dynamic mential structures

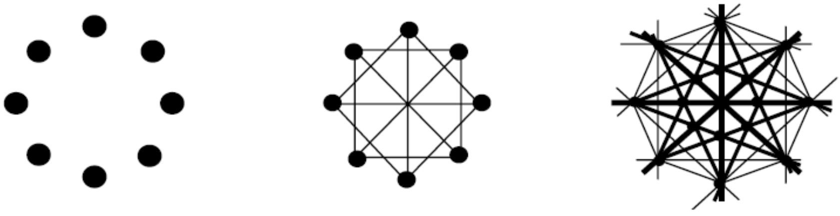
Typically, after studying a certain subject, the student (with the possible exception of the gifted student) rarely reaches the structural model of the assimilated information (that means the didactical model, because the model of that subject as a scientific domain is impossible to reach in school). The "output" then involves disconnected information, with inevitable discontinuities, difficult to be mobilized and used in new situations. This is, in fact, a failure from the very beginning in the case of those disciplines that naturally presuppose strong and explicit connections among concepts.

How do we avoid this situation? In principle, by "canceling the daily lesson", but this cannot be done, as school learning would become impossible. In practice, good teachers try to redo in new contexts what has been previously assimilated in order to establish a long-lasting connection with the new knowledge; however, this is done accidentally, unsystematically, and, as one gets along in the curriculum (or textbook), more and more of the essential information is left out. The process of returning to the basic information of the previous lessons is essential. It is not a question of repeating the previous essential information, but of assimilating bridges among the chunks of information in order to give the latter a certain stability and functional independence.

Any type of learning develops mental structures. Here, I use the term *mental structure* for both a conceptual entity belonging to a specific knowledge domain and for a mental representation generated by internalizing that entity. Consequently, the mental structures can be very different, having as referential: a concept/ procept, a group of concepts/ procepts bound together, an aggregate of substructures, a series of interdependent aggregates. The following three dimensions were considered to characterize a structure acquired through learning (Singer, 1995; 2001b):

(a) a *discrete component* representing the "nuclei" or the fixed, stable elements of the structure;
(b) a *contiguous component* that can be "visualized" as a network;
(c) a *cinematic component* representing the associations.

Taking into account the variation of these parameters, the following types of structures might be differentiated as distinct theoretical entities: rigid structures, flexible structures and dynamic structures. The schematic representations from Figure 4 give an intuitive insight into the differences between these three types of structures.



A rigid structure A flexible structure A dynamic structure

Figure 4. Schematic representations for different types of structures

Characteristics and examples of rigid structures

The evolvement of a rigid structure is a frequent phenomenon in learning. For example, it is known that children learning English at some point say “I runned down the street” instead of “I ran”, extending the regular form of the past tense to irregular verbs. We can consider that a rigid mental structure has led to this effect. Other examples are offered by the misleading learning of some skills such as playing an instrument or by the robust presence of misconceptions. Youngsters come to believe, for example, that one feels warm when wearing a sweater because the sweater itself generates warmth. The misconceived theories can be thought of as powerful engravings that have been incised upon the mind-brain of the child during the early years of life. The facts learned in school may seem to obscure these engravings; however, frequently, the initial erroneous engraving remains largely unaffected (Gardner, 1983, 1999). In these cases, the mind has generated rigid structures.

A *rigid structure* is characterized by: (a) oversized, very stable nucleus, (b) a poorly developed network, sometimes totally lacking, and (c) associations that function in the area of recognition of a standard situation and its reproduction. The phenomenon emerges frequently in learning classical geometry; the student recognizes the isosceles or right-angled triangle only if the given triangle is in a certain position; any other position is perceived as a new learning element that requires a new nucleus in the structure. Such rigid mental configurations often become fixations. Beyond its positive role in assuring the stability of the acquired knowledge, a rigid structure is usually responsible for the emergence of typical errors. Beyond the mind engravings, such a structure develops because of two types of errors in teaching. One error occurs when isolated information is taught without highlighting its connection with previously learned information or when insufficient time is given to the internalization processes needed to create a network. A second error occurs when excessive focus is placed on already-taught information. This, too, hinders the development of a network. Some of the “drill and practice” procedures that developed out of a behaviorist approach can be responsible for this result. A mental structure has a regenerative tendency to organize itself, a tendency that can be blocked only by the second above-mentioned constraint. In fact, the presence of that tendency explains the progress in learning even with the most inappropriate teaching.

Characteristics and examples of flexible structures

A more adaptive structure to a variable learning environment is a flexible structure. If each object we came across during an ordinary walk were new, then we would either give up all attempts at proceeding or we would stop every time to clear up each of those new objects, consequently there would be no progress. Even during the most unusual trips, the things we regard as new are not numerous. Thus, usually, we do not pay much attention to what we come across unless it is something really new to us, or it causes new problems (such as a tree fallen during the night, an acquaintance unusually dressed, something that raises our interest, etc.). A similar phenomenon happens while reading a book in our field of interest: we are less interested in what we already know; we only stop and ponder on new elements. The flexible structures activated in our minds acknowledge and allow us to integrate the new into a coherent context.

A *flexible structure* is characterized by: (*a*) stable nuclei, (*b*) a developed network, and by (*c*) associations based on recognizing invariant elements in various environments. A flexible structure allows problem solving through analogy and inductive or deductive inferences when the context is relatively familiar. Such a structure might enter into relations with other structures, ensuring a coherence of the reaction. Its optimality is expressed in daily informal learning. The acquisition of this type of structure might represent the ideal for learning school subjects such as history, geography, social sciences, etc. Nevertheless, such type of structure is not optimal for learning domains such as mathematics or languages, which are internally structured through complex networks of concepts and procedures in a hierarchical manner. In mathematics, merely solving ill defined real problems will not assure a learning behavior similar to that of an expert.

Characteristics and examples of dynamic structures

Some of us are able to rapidly see or discover connections among things that look to be not connected at a first sight; they can transfer mental tools developed in a certain context to analyze or solve a problem in a completely different context. Some of us can identify patterns where many of us see only disparate things. From a cognitive perspective, they have activated and mobilized dynamic mental structures that enable them to optimally respond to the situation. What specifically characterizes such a structure?

A *dynamic structure* implies: (*a*) flexible nuclei that are or could become structures themselves; (*b*) complex networks with ramifications and hierarchies; and (*c*) dynamic associations that facilitate quick mobilization of the structure through the discovery of critical paths. These associations can allow links between one structure and another, underlying the relations among these structures, as a whole or in part, and can stimulate self-development of the structure.

As stressed before, the main thesis of this article is that through adequate training and structured inputs, dynamic structures can be developed in children’s minds on a systematic basis. This seems to be a necessary condition for efficient problem-centered learning in mathematics. The dynamic mental structures activated through DSL could shorten the pathways to understanding by developing *generative connections* beyond the learned concept. A dynamic structure can also behave as either flexible or rigid, depending on the task to be solved. While the flexible structures are mostly adaptive, the dynamic ones are mostly creative.

The dynamics of mental structures depends on their kinematic training. This requires special attributes of mobility, which refer to:

- relating each nucleus of the structure to any of the others;
- mobilizing the whole structure on a given task in order to put new information into as many nuclei of the structure as possible, for the purpose to solve the task efficiently;
- making each nucleus of the structure flexible enough so that it might multiply or it might migrate into another structure;
- linking the structure to another nearby or distant structure;
- reorganizing the structure according to a certain working hypothesis while performing a task;
- re-constructing the whole structure starting from some of its nuclei or connections;
- transferring the structure from one abstraction level to another.

An example of a dynamic structure is the construction of the concept of recurrent sequence of natural numbers (presented in detail in Singer, 2007a). This construction uncovers the underlying connections between increasing and decreasing sequences and the four arithmetical operations. For example, the increasing sequences of even numbers (e.g. 0, 2 , 4 , 6, 8 , etc.) contains a synthesis of addition (e.g. 2 = 0+2, 4 =

2+2, 6 = 4+2, etc.) and multiplication (e.g. 6 = 2+2+2= 3x2, 8 = 4x2, etc.), while the decreasing one contains a synthesis of subtraction (e.g. 12 = 14 – 2, 10 = 12 – 2) and division (e.g. 12 is 6 times 2, then 12:2 = 6); this is why the sequence appears as an *integrating concept* of all the four arithmetical operations. Emphasizing these connections allows to set information in multiple nuclei in order to mobilize them for solving tasks or to reorganize the structure while performing a task, as previously presented. As the inclusion of arithmetical operations is not obviously visible, being *implicitly* integrated into the sequence, these operations get lost there if the support for learning is not adequate; this is actually happening in classical training. Within DSL, the notion of a sequence of natural numbers as an integrating concept is acquired in two ways: through processing within the frame of the sequence (identifying the rule involved in the specific pattern, adding new terms, applying a given rule when the first term is known, etc.) and by focusing on the “personality” of each element of the sequence. During the DSL, each element of the sequence is compared with the elements in the neighborhood (e.g. we get 8 by adding 2 to 6 or by subtracting 2 from 10), then with other elements (e.g. we get 8 by multiplying 4 by 2), and periodically, each element of the sequence is related to its concrete meaning (as the cardinal number of a specific set). In this way, the nuclei of the structure relate to others, making them flexible and transferable to various mathematical entities. According to the DSL theory, as described above, if the mechanism underlying sequences is internalized in grades 1 and 2, it generates an “expert behavior” that facilitates various categories of transfer: to any sequence of natural numbers; to connections among the arithmetical operations, simplifying the access to future understanding of the properties of operations and of “the priority rules” in computing chains of operations. Moreover, such a mechanism can be transferred to decimal numbers. Instead of learning the operations with decimal numbers as completely new entities – new separate nuclei – they can be learned as natural extensions of the algebraic operations with integers. Such a pattern is very complex and its formation triggers special teaching problems because it is necessary to train the mobility of concepts explicitly and systematically, and to shape their structure towards new ones, which are to be learned later.

DSL in practice: An insight into the teaching methodology

Any type of learning engenders mental structures. Initially, the mental configuration generated through learning is unstable; any new information can perturb it. Effective learning occurs if the new information comes into the existing configuration of information and completes it, i.e. if it produces contiguous connections, which generate acquisition of knowledge and skills with certain stability. However – and this is the critical point for training the mobility – if information transmission deepens the stability of the configuration, it becomes a negative factor. Classical training, excessively relying on memory for learning, prematurely strengthens the stability of a mental structure. At the opposite pole, the focus on only ill-defined real problems can generate a perception of mathematics as a trial-and-error domain. Maintaining a balance between the stability and the mobility of the mental structures generated by learning becomes a fundamental task of good pedagogy.

Structural cognitive learning aims at endowing children with domain specific competencies. These cannot be taught directly as such. They are structured gradually, as a result of a complex mental training. Are the models built within DSL sustainable in practice? The answer to this question supposes to develop and apply an adequate didactical technology and to assess its effectiveness in school contexts.

Evidence to support the assertion of the practical effectiveness of DSL was gathered in the following manner over a 4-year period: in year 1, two classes of children from a school in Bucharest (Romania) were taught according to the experimental methodology; two further classes were added in year 2; in year 3, five classes from the Republic of Moldova also joined the program. The number of children in the classes varied between 26 and 34 and, in total, 232 children in 9 experimental classes were involved. The experimental program tracked cohorts of children from grades 1 to 4 (aged 6-7 to 10-11). Preparatory meetings and feedback sessions were conducted with the classrooms teachers. The teachers received detailed description of the tasks that they were going to offer to students and the teaching periods were followed by discussions, on a weekly base. The description of the learning activities is contained in four teacher's guides (Singer and Radu, 1994-1997).

Constructing recurring cycles of understanding

To focus the ideas in accordance with the theory previously presented, I have chosen sample learning activities that lead to internalizing word problem solving involving addition and subtraction. According to DSL, the first step was to set up a classification of word problems that takes into account the mathematical categories that generate such problems. The need for taxonomy is given by the aim of focusing the training on the generators of problems, and not on a plethora of drill and practice techniques. This is actually the reason for which the DSL is conditioned by a careful curriculum development. Without going into details (they need more space than we have at our disposal in this paper), the specific training methodology for problem solving is based on highlighting the following aspects:

1. *Identifying the problem components.* To classify the elements of word problems, the following criteria were taken into account: type of theme, type of terms, type of action, and type of connection between terms and actions. The theme is offering the context for practicing union when using various categories of objects. The action is characterized by direction, sense and rhythm. Here the word "term" is used with a very wide meaning; for a child, it is about living beings that are moving by themselves – active - or about objects which are put together by somebody else - passive. The action can be explicit (external, visible, evident) or implicit (internal, non-visible). The connection between terms and actions can have different degrees of mobility from static to dynamic.

2. *Problem transposition.* A simple problem of addition (of the type $a+b = x$, x – the unknown) is proposed. The problem is reformulated keeping the same numbers (for example, by changing the position of the question). The problem is extended so as to contain two or three operations of addition, or subtraction or a combination of addition and subtraction.

3. *Problem comparison.* Constantly, a problem is compared with some others in order to uncover similarities and differences.

4. *Analyzing incomplete or redundant problems.* Problems that are missing data or other components are proposed. Children analyze if and why the solving is possible, and complete the text with additional information. Redundant problems are discussed, eliminating supplementary data that are not meaningful for the solving.

5. Generating word-problems. An important part of the training is focused on devising problems. This is done with starting support points and later without them. For example:

- *Starting from exercises.* A simple exercise is chosen and the task is to devise similar exercises. The starting exercise is compared with others (number of terms, operations) in order to highlight similarities and differences. Then the initial exercise is transformed into a problem. New proposals put forward exercises in which the number of terms and the number of operations are increased. For each exercise, a variety of word problems is generated keeping the same data, but changing the context.
- *Starting from symbolic schemes.* The teacher gives the scheme $a+b = x$ and requires examples of exercises. Then the teacher asks for the formulation of word problems. The position of the unknown is changed ($a+x = c$; $x = a+b$; $x+b = c$; etc.) with the same requirements (proposing exercises and problems). The same procedure is carried out starting from one of the other schemes of the type $a-b = x$, $a+b+c = x$, $a-b-c = x$, etc., or from graphical models, diagrams, tables.

6. Problem solving. A list of categories is used to classify addition/ subtraction word problems:

- *Grouping*. Example: “2 children are playing. 3 more children join the game. How many children are playing now?”. The mathematical model can be synthesized as:
  A, B sets, $A \cap B = \emptyset \quad \Rightarrow \quad \text{card }(A\cup B) = \text{card } A + \text{card } B$
- *Separating*. Example: “Ann has got 2 notebooks. How many more does she need in order to have 5?”. The mathematical model can be synthesized as:
  A, B sets, $A \cap B = \emptyset \quad \Rightarrow \quad \text{card } B = \text{card }(A\cup B) - \text{card } A$
- *Complementarization*. Examples: “A bouquet has 3 red flowers and 4 blue flowers. How many flowers are there in all?” “A bouquet has 7 flowers; 3 are red and the rest are blue. How many flowers are blue?”. The mathematical model can be synthesized as:
  $A \cup C_E A = E \quad \Rightarrow \quad \text{card } E = \text{card } A + \text{card } C_E A$
- *Comparing*. Example: “Ben has 6 stamps, and Ann has 5 more than Ben. How many stamps has Ann got?”. The mathematical model can be synthesized as:
  $A \subset B \quad \Rightarrow \quad \text{card } A < \text{card } B$
- *Equalizing*. Example: “There are 8 boys in a team. 2 more boys join them and now there are as many boys as girls. How many girls are there?” The mathematical models here suppose the use of equalities and equations.

Children are to practice the tasks succinctly described above in a gradual progression of internalizing that emphasize recurrent cycles of understanding: orally, silently (in the mind), in writing (without or with minimum verbalization and the result is required for checking). Letters are used just accidentally, or gradually, depending on the students’ level and teacher knowledge of their appropriate use.

During the school year, the children in the experimental classes were tested 15 times. Each test contained 10 items, amongst which 4 to 6 were open-ended questions requiring creative answers. The tests have been published as booklets (Singer and Raileanu, 1994-2000). The average success at solving creative tasks was more than 60% of students for each of the classes involved in the experiment. For a comparison, I refer to a national assessment in grade 4 at the end of the same school year, in which

18 844 students from 992 schools in urban and rural areas participated. In this assessment, an item requiring a non-standard creative answer was: “Devise a problem using numbers smaller than 20 that can be solved using addition, multiplication and subtraction”. This item was correctly solved by only 20% of the students (only the text of the word-problem was scored, not its solving), and almost half (46.1%) did not make any trial. This shows a habit of mind of avoiding creative problems, already in place in grade 4.

Conclusion

DSL as a pedagogy for school learning tries to move the didactical approach from a “horizontal” way of perceiving teaching – islands of information to be transmitted – to a permanent process of “vertically” restructuring students’ knowledge by incorporating the new elements into a dynamic structure. In particular, this implies systematically practicing the basic mental operations while creating patterns of variability. This kind of training presupposes identifying and developing optimal individual pathways in a multidimensional network. These pathways could be incorporated into a future approach to learning, one that not only *aims* at developing creative and fluent thinkers, but one that *is* also *able* to do so.

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