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Conceptual representations and versatile mathematical thinking

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Abstract

In this paper we will present one description of the important role of mathematical signs and representations in learning, exploring the links between these representations and mathematical schemas. In addition, we examine some of the learning problems that these representations pose for some students of mathematics, and how these might be addressed in teaching. In particular we consider the concept of mathematical versatility, and how versatile thinking might be encouraged. This involves a discussion of the qualitative nature of the interactions of learners with representations of processes and objects, and the links they form between different representations of concepts. Calculators with embedded computer algebra systems (CAS) are becoming more common in the learning of mathematics, and are increasingly seen as a means of providing access to new ways to thinking about mathematical concepts, including representational versatility. Hence, when the ideas presented are illustrated with examples from secondary school mathematics they often involve the use of CAS calculators.

Introduction

It should not be surprising that learning mathematics can be a complex process. There are so many variables involved, including important aspects associated with language and social interactions, psychological and epistemological factors, the cognitive and affective, the nature of mathematical concepts and their representations, and many others. However, whatever perspective we bring to the study of the variables we choose to interact with as mathematics educators, it is evident that at some stage in the learning process the learner has to engage personally with mathematical procedures and concepts. As they do so they will interact with these mathematical entities through a diverse range of external and internal representations that will be symbolised primarily linguistically, numerically and algebraically, but also figuratively, graphically, and in other ways. It is the role in learning of these representations, and the media in which they may be instantiated, that are the subject of this discussion.

While we have used the term representation in the above paragraph, its usage in the literature is not at all clear. The discipline of semiotics, however, can help us here. Semiotics involves the study of signs that bring to mind the object referred to (the referent) (Peirce, 1902), and these are variously described as icons, indexes, or symbols. While an icon resembles the object that it refers to, and an index has an embodied connection to its object, for a symbol the sign is not directly related by resemblance or physical connection to its referent. As Peirce (MS404, 1894) puts it a symbol is a sign that has become associated with its meaning by usage; it becomes significant simply by virtue of the fact that it will be so interpreted. Of course, mathematics predominantly comprises symbols, rather than icons or indexes. In turn, signs are often not isolated but are grouped together into families (Saussure, 1966). When signs are thus grouped together, for example in mathematics as logic signs, algebraic symbols, matrices, or graphs, then we shall call these *representation systems*, and the individual signs, representations. Thus a representation is a sign associated with a given system of signs. These systems are important since they give a context in which the sign may be interpreted. For example, the sign ην may refer to a composite function in a mathematical context, or to the word translated in English as 'was' in a Greek language context. According to Peirce's (1902) semiotic model we perceive a sign and recognise it as standing for something else, an object, and this in turn brings to mind the concept (in terms of the concept image—Tall and Vinner, 1981—the thoughts and images associated with the object in our minds). In a mathematics education context, Kaput (1987, p. 23) has described a representation as involving "two related but functionally separate entities… the representing world and the represented world". Symbolisation is referred to by Goldin (1987) as comprising precisely the correspondence and the relationships between these two worlds. We will now consider how the signs, objects and concepts are stored in the mind.

Since Piaget's pioneering work (e.g., Piaget, 1952, 1953; Piaget and Inhelder, 1969) it has been acknowledged that as individuals interact with their environment their mathematical learning will be influenced by their existing knowledge. While there is not universal agreement on what this knowledge comprises, or how it is stored or accessed, Piaget's notions of *schemes* and *schemas* have maintained considerable support among a wide range of researchers (see e.g., Davis and Tall, 2001; Lagrange, 2000; Vergnaud, 1999; von Glasersfeld, 1991). Piaget distinguished between schemes and schemas, and the difference to him was:

> "If we use the term 'scheme' (*schème*) to designate a generalization instrument enabling the subject to isolate and utilize the elements common to similar successive behaviours, then there are perceptual schemes, sensori-motor schemes, operational schemes, and so on. …But if we use the term 'schema' (*schèma*) to designate a simplified model intended to facilitate presentation (such as a topological schema, etc.), then there can be no perceptual 'schemata', since the 'schema' serves only for figuration and evocation." (Piaget and Inhelder, 1971, p. 366)

Likewise, some educators today speak about schemes as a regularised set of actions to perform a particular task (Lagrange, 1999). In the context of learning, schemas have been given a number of interpretations in the literature. For example, Skemp (1985) describes how we construct 'what we already know' by engaging in mental construction of reality by building and testing a schematic knowledge structure, where a schema is "a conceptual structure existing in its own right, independently of action" (Skemp, 1979, p. 219). In the context of problem solving, Paas (1992, p. 429) describes how a schema "can be conceptualised as a cognitive structure that enables problem solvers to recognize problems as belonging to a particular category of problems that require particular operations to reach a solution", while Sweller (1992, p. 47) defines a schema as "a cognitive construct that permits problem solvers to categorise problems according to the moves required to solve them."

Our existing schemas either promote or restrict the association of new concepts through assimilation, and so the quality of what an individual already knows is a key determinant of ability to understand, and so "our conceptual structures are a major factor of our progress" (Skemp, 1979, p. 113). Thus the richer an individual's schemas are the greater his/her ability to interact meaningfully with representations of mathematical entities. What are the variables that determine the richness of schemas? According to Anderson (1995) conceptual growth and understanding can be interpreted in terms of conceptual nodes and relations between nodes. He describes (*ibid*) the spread of the network and the strength of the links between the various components of information located within the network as variables determining the quality of a schema. In addition, for Skemp (1979) the qualitative nature of the links is vital. He describes them as associative (or rote learned) or conceptual. Thus the richness of conceptual schemas in a given knowledge domain, measured in terms of variables such as the spread of the network, the qualitative nature of the links between the constituents, and the strength of these links, promotes assimilation of new knowledge and consequent expansion of the schemas.

Another perspective on the content growth of schemas is provided by Dubinsky and others (Dubinsky, 1991, Cottrill *et al.*, 1996), who use the acronym APOS to describe the four components of *Action*, *Process*, *Object*, *Schema* in the building of mathematical knowledge. The chain of events, they suggest, develops as follows. Repeated actions, when applied to objects become internalised as processes, which in turn become *encapsulated* as mental objects. In turn examples of these three link together to form cognitive structures or schemas. Thus, conceptual entities in mathematics often present themselves with two distinct but complementary faces; they may be viewed as dynamic processes or as static objects. To make a mathematical idea readily manipulable and applicable in other contexts, it must be available internally in a concise form and the encapsulation of the process as an object is one way of accomplishing this. The relations that are constructed between the conceptual objects forming a schema could represent, for example: similarities and dissimilarities between concepts; instances of a concept; procedures for using concepts for solving problems; or affective factors related to those concepts.

These objects themselves arise through interaction with other process/object entities, either by different levels of observation (or perception), or physical and mental actions that lead to the abstraction of properties (Sfard, 1991; Gray and Tall, 1994; Tall, Thomas, Davis, Gray, and Simpson, 2000; Thomas, 2001; Thomas and Hong, 2001). In turn the mathematical objects are symbolised, and this sign may have as its referent either the process or the object, and has been called a procept (Gray and Tall, 1994). For example a process may be encapsulated as an object, as in the case of algebraic expressions and Riemann sums. These objects may themselves then be the subject of further observation or actions, and so the schematic structure develops.

Turning attention to the ways in which students interact with representation or notation systems, Kaput (1989) draws a distinction between the way some are used mainly to display information and relationships *(display notations)* while others support a variety of transformations and other actions on their objects *(action notations)*. Later he described an important class of mathematical activity as involving "translations between notation systems, including the coordination of action across notation systems." (Kaput, 1992, p. 524), explaining the importance of enabling manipulation of mathematical concepts both within and between these different representations. Thus it is important to recognise that understanding of a mathematical concept should include the ability to relate corresponding elements of different representations of the concept. In the language of semiotics, this relationship is called a semiotic morphism. However, the establishment of this morphism, or correspondence, is not free from ambiguity (Goldin, 1987).

Thus concepts, the meaning behind the mathematical constructs, do not usually arise from a particular representation system, but are the product of both individual and societal experience of interaction with a number of related, interacting representational systems, with each emphasising and de-emphasising different characteristics of the constructs (Lesh, 1999, 2000). This is discussed in more detail below.

Interactions with representations

In the light of the above we may see that when an individual or a group of individuals interact with a given mathematical sign or family of signs these are interpreted by their association with mental schemas. The sign is part of a representational system and its referent sits within a schema. In turn what we learn from the sign or representation will be influenced by the schema and by the content of the representation. Consider for example the representations of the earth shown in Figure 1. Figure 1a comprises a version of the common Mercator's (cylindrical conformal) projection. While the projection is valuable for navigators because it preserves angles, it is well known, although we can easily forget, that this projection distorts certain facts about the earth. For example, Africa may look smaller than it is in reality. Figures 1b, c (taken from Delmelle, 2001) show two other projections (from many that are possible), namely the Mollwede equal area projection and Canter's projection minimising distance proportions. Interacting with these representations will give us better data about relative areas of countries or relative distances between places, respectively. Our concept of the earth will be refined the more representations we can interact with. Of course to gain the most from these representations one should have schemas that include knowledge of projections such as regular, transverse and oblique cylindrical, polar and oblique azimuthal (planar) and regular conic. The final projection in Figure 1 (1d) is a favourite of those from 'down under' New Zealand, since it shows the earth centred on Wellington, New Zealand. However, we can all learn something from interacting with a representation such as this one. For example, the dark circle represents a hemisphere (about 11 hours flying time from Wellington). Looking at what lies within this circle may help one to appreciate things such as a) how large the Pacific ocean really is, b) how far New Zealand is from most other parts of the world, and c) how big Antarctica is, namely larger than Europe and 1.69 times the size of Australia, even without its ice shelves.

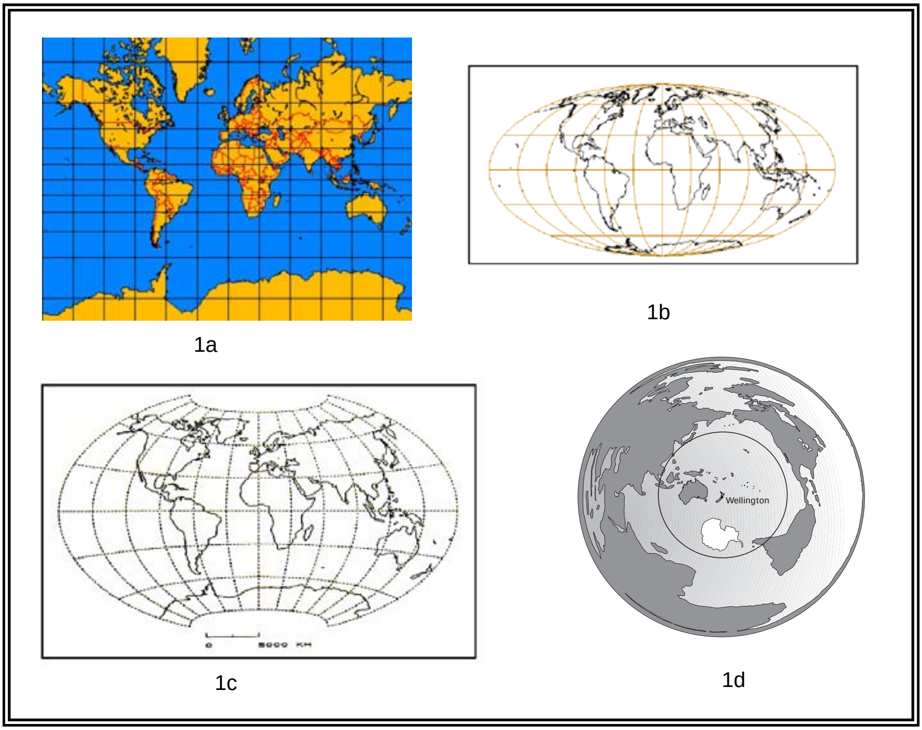


Figure 1. Four different projective representations of the earth.

A second example will illustrate that sometimes a representational system produces cognitive conflict when we interact with examples of its signs or representations. When we consider the symbols used to represent monetary amounts, we may use $1.17 or 38.5 cents without any problem. However, there is evidence that in some situations these two different positions of the decimal point cause conflict. Recent observation of the pricing of petrol (gas) on garage forecourts confirms this. Figure 2 shows some of the results of the conflict.

Figure 2a shows a reluctance when using dollars to write 1.069 or 1.119, and instead 1.069 is used. This is followed through in the second example (2b), where in using cents, 105.9 and 110.9, the 9 is treated in a similar manner. Figures 2c and 2d show just how demanding coping with these two systems of decimals can be. The final Figure (2e) shows that it can be done successfully with the figures in cents, although I have yet to see 1.139 dollars (note that petrol prices have risen here too in line with the rest of the world in recent times!). While there may be other reasons for this apparent inability to combine two decimal representations, such as a desire to hide the 0.9 cents, talking to the petrol station attendants it was clear that there was a conceptual problem too.

Figure 2a

Figure 2b

Figure 2c

Figure 2d

Figure 2e

Figure 2. Petrol prices showing problems coping with decimal representations.

Other mathematical examples can be used to show that what we learn can be representation-dependent. Consider the case of how we learn to decide whether a number is even or not. Most people will quickly say that if the number ends in a multiple of 2 then it is even. So numbers such as 12, 38, and 54 will be pronounced even. However, they may fail to appreciate that this is a feature of the common base ten representation rather than the even number concept itself, since 12_3, 38_9, 54_7, 32_5, 31_4, and 45_6 are all odd, although 22_3, 48_9, 64_7, 42_6 are even, and so are 11_3, 35_9, 53_7, 31_5. This seems to suggest that, in the words of Peirce, the iconic nature of the sign may be beginning to take over from its symbolic role, thus blurring features of the concept in the mind of the beholder. This may also be true at higher levels of mathematics where a graphical representation acting as a sign may take on an iconic nature rather than a symbolic one. In this case students may look at a graph and declare it to be the graph of a continuous function "because it looks continuous", or they may confuse the variables underlying a graph, for example leaning toward an embodied view of it as representing the physical movement of an object up and down, instead of its velocity. Here the graph tends to become the object in place of a function referent.

Another problem with new signs or representations is that because, for the individual, they do not naturally fall into a particular representational system they may not be easily assimilated into a schema, and thus have no 'natural' referent or concept. In this case the mind will often associate the sign with what appears to be the closest appropriate schema. This may be illustrated by the sign or representation in Figure 3. Here the problem given is to find the value of the number that has been replaced by the ?.

A	A	A	A	28
A	A	B	B	30
B	C	D	A	20
D	D	C	B	16
?	19	20	30	

Figure 3. A sign made of letter and numbers in a grid.

Some may find this closest to belonging to the matrix representational system, but for many the most logical mathematical schema seems to be that of the algebra of generalised arithmetic, and almost before they know it they may trying to find the value of ? by constructing equations such as 4A=28, 2A+2B=30, or 2A+C+D=19 and trying to solve them for A, B, C and D. However, such a schematic link obscures the fact that there is a much more appropriate method of solving the problem using a procedure arising from a property of the sign itself. Since each letter in the square is counted twice, once in the across sums, and once in the down sums, then it should be clear that the total of the across sums = the total of the down sums. So ?+19+20+30 = 28+30+20+16 and ?=25. Hence this is an example of the referent of the sub-signs that are a constituent part of the representation (here, letter as variable), exerting an undue influence on our mathematical thinking. They form a strong link to a schema, thus making the problem harder than it should be. A similar situation may exist with graphs of functions. As mentioned above, when deciding on continuity students may use the graphical representation and apply the maxim that if it looks continuous then it is. This may not be true, and requires special attention when the graph is in a technological medium.

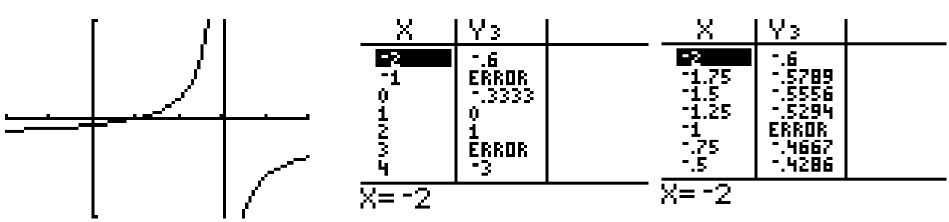


Figure 4. Examining continuity by linking representations.

Figure 4 gives the example of a consideration of the continuity of the function

$$f(x) = \frac{1-x^2}{x^2-2x-3}$$

at the point $x = -1$. Here the problem at $x = -1$ is not apparent using the calculator's graphical representation. However, the limit

$$\lim_{x \to -1} \frac{1-x^2}{x^2-2x-3}$$

may be considered via a graphic calculator (GC) technique of 'zooming in' using the alternative table representation, which is linked to the graph in the calculator, as shown.

If a new sign or representation is not readily assimilable by a student into a schema then a problem may arise in its interpretation. This may be due to any of the problems outlined above, or for some other reason. In this case the individual will try to make sense of the sign by assimilating it into some mathematical schema. The result may be the kind of thinking that Vinner (1997) describes as *pseudo-conceptual*. He explains that such thinking exhibits "behaviour which might look like conceptual behaviour, but which in fact is produced by mental processes which do not characterize conceptual behaviour." (*ibid*, p. 100). In a similar manner I have previously described (Thomas, 2002) how occasionally students may appear to have encapsulated processes as objects but that this turns out on inspection to be what I term a *pseudo-encapsulation*. I first described an example of this in 1986 (see Thomas, 1988; Tall and Thomas, 1991) on discovering that 47% of a sample of 13 year-old students thought that 6÷7 and $\frac{6}{7}$ were not the same, because, according to them, the first was a 'sum' but the second was a 'fraction'. Such students had not encapsulated division of integers as fractions. Instead they had constructed a *pseudo-encapsulation* of fraction, mostly thinking of it as 'parts out of'.

At a higher level in school many students seem to construct a *pseudo-encapsulation* of derivatives represented by signs such as *dy/dx* (see delos Santos and Thomas, 2001, 2003). Students may appear to deal with this as if it is the derivative object, the encapsulation of the process of differentiation, but often for them it is not. Instead the sign simply refers to carrying out the process of differentiation (or even just applying a known procedure for it). This is exemplified by research (delos Santos and Thomas, 2001) that describes how a considerable number of students who were able to understand *dy/dx* as standing for rate of change or derivative in

$$\frac{dy}{dx} = 5x$$

were unable to see it the same way in the form

$$2x + \frac{dy}{dx} = 1\cdot$$

If such students were seeing *dy/dx* as an action or process with a resulting expression then this could explain the difficulty, since the symbol, representing the process would have to be on the left and the result on the right, analogous to Kieran's (1981) description of children's view of the equal sign. It is much more difficult to see the second equation in this way. Such a *process-oriented* perspective (Thomas, 1994) on the sign *dy/dx* appears to be at the root of the difficulty students often have with perceiving the equivalence of the referent of the signs

$$\frac{d\left(\frac{dy}{dx}\right)}{dx} \text{ and } \frac{d^2y}{dx^2}.$$

They tend to understand the latter as ‘do the process of differentiation twice’, but the former has little meaning since it requires the object *dy/dx* to be operated on, something that a pseudo-encapsulation perspective does not allow, since it does not produce a true mathematical object.

The role of interpretation of signs

One of the major difficulties dealing with mathematical signs is that an association with their referent often requires considerable work on the part of the beholder. This is true even of the iconic signs of geometry, such as the rectangle sign shown below.

Laborde (1993a, b) has explained how one may see such an icon in two different ways, which may be described as surface and deep observations. She talks about how a *Drawing* refers to the material entity while *figure* refers to the theoretical object (Laborde, 1993b, p. 49). Thus a surface observation of this icon may lead us to think that it may be a representation of a mathematical object, but in order to move from this idea to seeing it as a figure, as referring directly to the mathematical object itself, requires interpretation. Laborde (1993b, p. 66) comments on how a program such as Cabri may contribute to students’ gaining of such interpretation, saying “At a low level the figure is viewed as an entity but not analysed into parts or elements: all parts of the drawing must move together. . . ”. This interpretation involves the use of an appropriate, existing mathematical schema to ascertain the properties of a rectangle that may be overlaid in memory on the mental representation of the drawing. Thus the concept of rectangle, is a combination of the perceived icon, its object referent and data (properties) from an appropriate mathematical schema. With many signs this process may have more than one stage. As I have attempted to explain elsewhere, (see Booth and Thomas, 2000), in order to produce a mathematical figure from a picture of reality we need to pay attention to the essential property revealing details of the picture in two steps. First we mentally or physically produce a diagram or figure from the picture, and secondly we need to overlay its conceptual properties in order to see the figure as representing the theoretical object (see Figure 5).

Perceived reality	◄—► interpret	Picture of reality	◄—► interpret	Diagram or drawing	◄—► interpret	Theoretical mathematical figure

Figure 5. Interpretation needed to convert a picture to a mathematical figure.

Booth and Thomas (2000) found that 13/14 year-old students with mathematics difficulties differed in their ability to solve problems depending on whether they were given a picture or a diagram prompt (see Figure 6 for an example of the questions). They were all better able to use the diagram than the picture and this may be due to the added interpretation needed to abstract the relevant properties to turn the picture into a useful diagram (for example we need to interpret the fact that the trees are assumed to be at the very ends of the path). This interpreting mode corresponds to what Mason (1992, p. 5) calls *looking through* images, since:

> .” . . to get much educational benefit, students need to be active in processing images; they need to *work on* the images, not just *look at* them. . . The point about processing diagrams is that it is not just a matter of *looking at* them, but rather *looking through* them.”

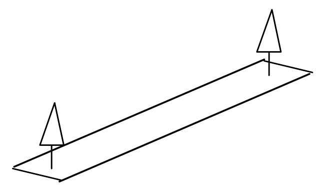
Picture Condition: The length of the path is now 7 metres. He planted a tree at the two ends and then a tree every metre along. How many trees were planted this time?	
Diagram Condition: The length of the path is now 10 metres. He planted a tree at the two ends and then a tree every 2 metres along. How many trees were planted this time?	

Figure 6. Picture and diagram forms of corresponding questions.

Encouraging a versatile approach

What kind of experiences then might help students in their construction of the mathematical concepts lying behind the symbols used to represent them? What will help them to interpret the symbols in a way that refers them to appropriate mathematical concepts? One idea that I have been proposing for some time (see Thomas, 1988) is that of *versatility* of mathematical thinking. *Versatile mathematical thinking* as I now enunciate it has three key features:

> *process/object versatility*—the ability to switch at will in any given representational system between a perception of a mathematical entity as a process or an object;
>
> *visuo/analytic versatility*—the ability to exploit the power of visual schemas by linking them to relevant logico/analytic schemas;
>
> *representational versatility*—the ability to work seamlessly within and between representations, and to engage in procedural and conceptual interactions with representations.

I will briefly address each of these and their benefits to learning.

Process/object and visuo/analytic versatility

I have already written above about how an individual *encapsulates* a mathematical process so that it becomes for them an object which can be symbolised as a *procept* and so I will not spend much time on it here. However, it seems that mathematics teaching in schools has paid very much less attention to *object-oriented* processes, those from which mathematical objects are encapsulated, than it has to *solution-oriented* processes, which are essentially algorithms directed at solving 'standard' mathematical problems (Thomas, 2002). An example of the first might be the addition of terms to find the partial sums of a series, which leads to the conceptual object of limit, and the second could be exemplified by procedures to solve linear algebraic equations. However *solution-oriented* processes usually operate on the very objects that arise from previously experienced *object-oriented* processes (for example students may be taught to solve linear algebraic equations, which requires them to operate on the variables, before they have encapsulated the variable object). Hence ignoring these is short sighted and results indicate that this tendency proves counter-productive in the long term. Failure to give students the opportunity to encapsulate *object-oriented* processes as objects may lead them to engage in something similar to the *pseudo-conceptual* thinking described by Vinner (1997).

To be successful in learning mathematics requires much more than the ability to carry out a succession of discrete solution steps. An overall picture of the task at hand is needed, so that the appropriate solution path can be selected and any errors that occur are more likely to be sensed and corrected. Thus a sequential/logical/analytic approach to mathematical processes should be complemented by a global/holistic overall grasp of the context. It is here that the visuo/analytic versatility is of value. The differences between these forms of thinking has been a focus of thought over the centuries. Descartes (1628), for example, contrasted the intuition of an immediate perception of connections between concepts with chains of logical deduction required to give formal relationships, and Poincaré (1903) distinguished between mathematicians who thought in a predominantly sequential/ deductive mode, and those whose work developed more through intuition. More recently Krutetskii (1976) divided his mathematically gifted pupils into analytic, geometric and harmonic types, according to their preferences for verbal-logical, visual-spatial, or a combination of the two. Brumby (1982) had a similar division, referring to those with a combined approach as versatile students. For some I have time proposed a model of *cognitive integration* (see Thomas, 1988, 1994; Thomas, 2002; Booth and Thomas, 2000) that stresses the value of utilising the two integrated, but qualitatively different schemas, and their corresponding modes of thinking, one of which is visual and holistic, and unconscious. If each is actively promoted in teaching then the complementary modes are both made available to the mathematical thinking of the student, and they may develop a versatility of thinking, moving freely and easily between them when the mathematical situation renders it appropriate.

Representational versatility

As explained above, meaning attributed to mathematical concepts is constructed from properties distributed across a number of symbols in different representational systems. Thus, in order to build a meaningful version of the concept, it is essential that the links between the representations are addressed, making the provision of multiple representations valuable for improving the capacity for learning (Kaput, 1992; Noss and Hoyles, 1996). This ability to establish meaningful links between and among representational forms and to translate meaning from one representation to another has been referred to as *representational fluency* (Lesh, 1999), or as *representational competence* (Shafrir, 1999). Thomas and Hong (2001) introduced the concept of *representational versatility* to include both this fluency of translation between representations, and the ability to interact procedurally and conceptually with individual representations (see below for more details on interactions).

Even (1998, p. 106) gives a very nice example of the power of representational fluency in the following question: "If you substitute 1 for x in $ax^2 + bx + c$, (a, b, and c are real numbers), you get a positive number. Substituting 6 gives a negative number. How many real solutions does the equation $ax^2 + bx + c = 0$ have? Explain." She reports that only 14% of 152 college mathematics students solved this problem. For 80% of the students the algebraic representation dominated their thinking; they only considered it, and did not solve the problem, which is rather tricky in this mode. However, those who were successful linked the given algebra to the graph of the function and used (without necessarily specifying it) the Intermediate Value Theorem, and 18 of them solved it. An example such as this helps use understand why Lesh (2000, p. 74) has suggested that the idea of representational fluency is "at the heart of what it means to 'understand' many of the more important underlying mathematical constructs" Likewise, Moshkovitch, Schoenfeld and Arcavi (1993, p. 97) suggest that we should ask "Can the student move flexibly across representations and perspectives when the task warrants it?…Does any curriculum we propose make adequate connections across representations and perspectives? If not it had better be revised". Based on the results of a number of studies (see e.g., delos Santos and Thomas, 2001; Hong and Thomas, 1997, 1998, 2001; Hong, Thomas, and Kwon, 2000), I would concur. However, this ability is often lacking in our students. Figure 7 shows what we found to be a surprisingly rare example of a student (taken from Hong, Thomas and Kwon, 2000) who is able to solve a linear algebraic equation using a discrete tabular representation of the functions in the equation.

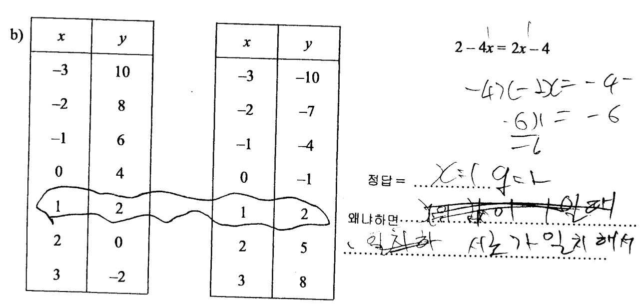


Figure 7. Inter-representational thinking in solving a linear algebraic equation.

The second aspect of representational versatility is the ability to undertake versatile interactions with representations. Thomas and Hong (2001) have divided interactions with representations into two types: observation and action. They further subdivide these into procedural, process and object interactions. The key point of their description is that we learn to abstract properties that lead to mathematical concepts by either surface or deep observation (*looking at* or *looking through* in the words of Mason, 1992), or by considering the results of our actions on the representations of the objects. However, our actions may be qualitatively limited by our existing perceptions of the mathematical objects. Thomas and Hong (2001) propose that a crucial difference between a process and object interaction is that the former may be approached discretely in terms of its parts (e.g pointwise for a function) while the latter requires a holistic perspective. This is the difference between being able to estimate (from a graphical representation) or calculate (from an algebraic representation) the gradient at a number of values of a function, and being able to use the derived function as the encapsulated object that (potentially) gives all such values.

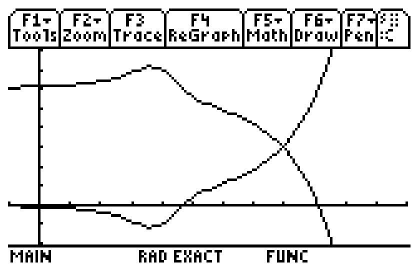


Figure 8. What is the relationship between the functions whose graphs are given?

If we are asked to express algebraically the relationship between the two functions whose graphs are reflections of each other in the line $y = 3$ (see Figure 8), without being given an algebraic representation of either individual function, then we may approach the problem in a process manner, by looking at what happens to individual points, or by considering the graph object as a whole. Representational fluency, of course, also comes into play here since we need to symbolise the relationship algebraically in order to describe it clearly.

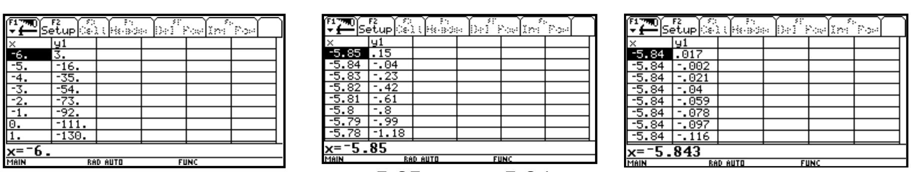
An example of how interactions with representations might be widened is the solution of linear algebraic equations. Students learn (although sadly not nearly enough of them) how to solve for a variable, say *x*, in an equation of the form $ax + b = cx + d$ by adding a constant or a constant multiple of the variable to both sides. This becomes for many a procedural interaction with the algebraic representation. However, they may not construct the concept of conservation of equivalence of equations under such operations. As explained in Hong, Thomas and Kwon (2000) it might be better if they interacted with these equations using ***any*** *legitimate transformation* (e.g. adding $\pm kx$ or $\pm k$, any real *k*, to both sides) learning that these conserve the solution, before moving to the use of a *productive transformation* (e.g. adding $-cx$ or $-d$ to both sides) that will take them more directly to that solution.

Conclusion

The examples above illustrate that some technology use may assist with improving the quality of interactions with representations. It may enable students to interact with concept representations in ways that would be much more difficult without it. In particular, portable technology such as graphic or computer algebra system (CAS) calculators have a number of advantages (see also Thomas, Monaghan and Pierce, 2004; Thomas and Holton, 2003). They may: encourage inter-representational thinking; enable new types of representations; enable new types of interactions with representations; enable access to representations that may challenge understanding; make conceptual investigation and generalisation more amenable. For example, Graham and Thomas (2000) used graphic calculators to assist students with construction of the concept of variable. They provided a link between an algebraic representation using letters and a model of a store with a label. The GCs enabled access to a new type of representations for the students, comprising multiple lists of the letter labels and the store contents, as shown in Figure 9. Interacting with the representation in a conceptual way, changing the contents of the stores and their labels, enabled investigation of the concept, and helped students build a better conception of variable.

Figure 9. Graphic calculator representations of operations with variables.

Novel interactions that such technology makes available include the ability to solve equations by numerical processes such as zooming in on solutions using tables of values (see Figure 10). They also enable new techniques, such as the use of $\int_a^b |f(x)| dx$ to calculate areas under graphs of functions (see Hong and Thomas, 2004), such as $\int_0^\pi |\cos x| dx$ without having to worry about zeros in the integration range. These techniques include the use of numerical methods in a graphical mode.



$-6 < x < -5$ $-5.85 < x < -5.84$ $-5.843 < x < -5.842$

Figure 10. A new calculator interaction with a tabular representations.

In conclusion we might say that because conceptual ideas may be constructed from a number of representations it is a good idea for students to experience a number of these at the time they begin to learn the concept. In particular, the explicit linking of ideas across representations is very useful and important. Hence, teaching should seek to assist representational versatility by concurrently providing, and linking, a number of representations in each learning situation. Further, since student interactions with representations can vary qualitatively, they should be exposed to both procedural and conceptual interactions with the representations. Technology such as CAS may be used open up possibilities both for new kinds of representations and new representational interactions. However, since procedures are often representation-dependant too, students will need to develop new procedures, new techniques (Lagrange, 2000) and new interpretations of outcomes.

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