Early algebra: From teacher education to classroom culture

Nicolina A. Malara
University of Modena and Reggio Emilia, Italy

ICME 10 2004

RL

Regular Lecture

Abstract

After a short survey of the main steps which led to the birth of early a algebra as a specific field of research, we sketch the main features of our Project (ArAl), conceived to promote algebraic thinking, which is also a teachers in-service education process. We describe the methodology of our work with and for the teachers, aimed at leading them to an awareness of the cultural choices of the project and, at the same time, at putting into practice constructive paths for the teaching of early algebra. We then present some excerpts of a didactical process, realized in third-grade classes, which are meaningful from the point of view of pupils' productions and of the classroom culture induced by the teacher. We conclude with some general considerations about the attitudes developed by the pupils and the teachers involved in the project.

On early algebra

The dawn of *early algebra* as a field of study dates back to the late eighties, when some wide-scoped surveys on the difficulties of learning algebra (see for instance Kieran, 1989) exposed the negative impact of traditional methods for the teaching of arithmetic, which were essentially based on aspects of calculation, paying little attention to the relational and structural aspects of arithmetic.

In a critical review of Kieran's study published in Wagner and Kieran (1989), Both writes:

> "Students'difficulties in algebra, it has generally been assumed, are largely difficulties in learning the syntax. Over the past decade, however, research evidence has been accumulating to indicate that many students have a very poor understanding of the relations and mathematical structures that are the basis of algebraic representation. This lack of understanding is not a new "algebraic" phenomenon: the research summarized by Kieran shows that the problem has its origin in arithmetic. Indeed, a major part of students' difficulties in algebra stems precisely from their lack of understanding of arithmetical relations. The ability to work meaningfully in algebra, and thereby handle the notational conventions with ease, requires that students first develop a semantic understanding of arithmetic. One task for research is to examine the whole question of students'recognition and use of structure and how this recognition may develop. A second task is to use this information to devise new learning activities and environments to assist students in this development. …"

In the same book Herscovics highlights the fact that many cognitive obstacles are of a historical-epistemological or psychological nature. On the subject of the latter, he writes:

> "From the piagetian perspective, the acquisition of knowledge is a process involving a constant interaction between the learning subject and his or her environment. This process of equilibration involves not only assimilation – the integration of the things to be known into some existing cognitive structure – but also accommodation – changes in the learner's cognitive structure necessitated by the acquisition of new knowledge (p. 62). …
>
> The obstacles associated with the learner's process of accomodation are pedagogically the most challenging. … What kind of pedagogical intervention can help the process along? … Problems need to be presented that can be understood by the learners but which cannot be solved within their existing knowledge (or at least not readily solved). Having created the need for change, the new matherial has to be organized into a constructivistic teaching sequence, that is, a sequence starting from the learner's condition and expanding from it. … For any mathematical concept that is new to learners, the best we can do is create conditions likely to enable them to complete the difficult process of accomodation. There are, however, no guaranteed recipies. (p. 83)"

Also to the same period belongs the debate on what may be, at curricular level, the ideal school age at which to introduce an initiation to algebra. Authoritative scholars (e.g. Usiskin, 1987) maintain that this should happen in the eighth grade, while others provide documentation of curricular experiments conducted in their respective countries, with the introduction of algebra in the sixth grade (Pegg and Redden, 1990).

Those years also saw the creation of interesting projects for the teaching of mathematics (in grades 6 to 10), which promoted a constructive approach to algebra from the very start, emphasizing the observation of relations and introducing the use of letters as a synthesis tool for analogous numerical relations or for the formal translation of verbal expressions which describe observed relations. Interpretative activities on formulae were also promoted, through comparisons between and reflections on different writings representting the same thing and similar writings having different meanings. There was an appreciation of methods based on reasoned attempts at the solution of equations and inequalities (see for instance Bell et al., 1985; Harper, 1987).

A significant milestone in this evolution process was the debate within the primary-school group of the working group on algebra at ICME 7 (Québec, 1992), co-ordinated by L. Linchevski (1995). In that context, it was emphasised that many of the difficulties pupils encounter in learning algebra are caused by the mainly procedural teaching of arithmetic in primary school, which leads to an inevitable cognitive gap in the transition to algebra, given the relational and structural aspects which are emphasised in algebra and it was highlighted that "within primary-school arithmetic there is ample opportunity for the development of algebraic thought" and that "letters could be used within children arithmetic experience in order to facilitate their understanding of the meaning and significance of letters in later, formal algebra".

More particularly, with Lincheviski and Sfard's theory of reification (1994) in the background, a new area in the teaching of arithmetic - called *pre-algebra* - is conceived, with the aim of developing 'pre-concepts' useful in algebra, that is, complex arithmetical concepts of a structural type, representing the experience and conceptual basis into which to introduce more abstract and formal algebraic concepts, which can help overcome traditional didactical cuts (Filloy, 1990) or 'cognitive gaps' (Herscovics and Linchevski,1994).

In that context the group suggested a set of activities belonging to the pre-algebraic field, which included, among others: a) activities aimed at seeing the particular within the general and promoting generalization processes; b) the detection of analogies or differences within the structure of arithmetical expressions, through the analysis of the represented calculation processes and the highlighting of the role of brackets; c) the ingenuous solution of equations, giving ample space to numerical substitution strategies and reasoned trials, aiming at the creation of appropriate cognitive schemes, through reflections on activated strategies; d) an introduction to the solution of verbal algebraic problems, through explorative procedures that bridge the gap between arithmetical and algebraic methods (for instance, those of 'false position').

In those years, several scholars highlighted the importance that pupils should acquire the 'sense of the symbol' through varied activities that help mature their abilities, comprehension, and different ways of feeling, leading them to acting with flexibility and instinctively in relation to a given set of symbols, to moving through wider or different systems of symbols and to co-ordinating various interpretations of formulae in different risolutive worlds (Arcavi, 1994; Arzarello et al., 1993; Gray and Tall, 1993; Filloy, 1990; Kaput, 1991; Lins, 1990). Also the USA starts proposing and debating the algebraization of the K-12 curriculum (Kaput 1995, quoted by Carraher, 2001).

The second half of the nineties saw a flourishing of studies on these aspects – both theoretical and experimental – and mainly targeted at pupils aged 11 to 13 (e.g. Ainley, 1999; Brito Lima and Da Rocha Falcão, 1997; Da Rocha Falcão et al., 2000; Boulton Lewis et al., 1998; Carraher et al., 2000). Some of the studies stand out for their theorising of models for a conceptual development in algebra of a socio-constructive type, which highlights the influence of the classroom environment on teaching and promotes the use of physical means as tools of semiotic mediation – all within the framework of an algebraic vision of algebra as a language (e.g. Da Rocha Falcão, 1995; Meira, 1990, 1996; Radford and Grenier, 1996).

In an analytical study of students' algebraic notations on the use of the scales as a mediation tool for making algebraic equivalences meaningful, Meira (1996) writes:

> "Using notations is obviously part of the very complex process of thinking algebraically (sometime an unnecessary part), but that indicates and supports the individual' insertion in certain discursive practices that are critical for one's participation and access to mathematics and, in particular, to algebra. In this respect, using algebraic notations as part of a language connects the individual to 'the spoken language of the mathematics classrooms' to 'the use of particular words for mathematical ends'; to 'the language of [mathematical] texts'; and also to 'the language of written symbolic forms'. It is critical to note that this view does not limit algebra to the use of its notational system, nor to algebraic activity to the meaning intended by experts. The situation is similar to the young child that mumbles words in very simple sentences without completeness or syntactical correction, but plenty of meaning for the communication being attempted with an adult or a peer".

Later, Radford (2000), unifying within a wide theoretical framework historico-epistemological, psychological, semeiotic and didactic studies, goes as far as to describe the learning of algebra as "the appropriation of a new and specific mathematical way of acting and thinking which is dialectically interwoven with a novel use and production of signs whose meanings are acquired by the students as a result of their social immersion into mathematical activities". He sees "the signs as tools or prostheses of the mind to accomplish actions as required by the contextual activities in which the individuals engage", shifting the focus "from what signs represent to what they 'enable' us to do". Moreover, he underlines the key role of the teacher in establishing the mathematical practice in a social context.

The transposition of these conceptions from theory into practice – also motivated by the presence of early algebra in the school curricula of countries such as the UK (1991) and the USA (2000) – poses the problem of teacher training and leads to the creation of specific innovation projects, as documented by the forum on early algebra within PME (2001) and more extensively by the Early Algebra section of the 12th ICMI Study, "The future of the teaching and learning of algebra" (Chick et al., 2001).

To this framework belongs also our own *ArAl Project, Arithmetic pathways to favour pre-algebraic thinking* (cf. Malara and Navarra, 2000, 2001, 2003a, 2003b)[1], a project born in 1998 on the basis of our previous studies at middle school level (Malara ,1994; Malara and Iaderosa, 1999), and subsequently conceived for primary schools within the perspective of continuity between the two school levels.

The hypothetical premise of the ArAl project

The basic premise of the ArAl Project is the existence of an analogy between the learning modalities of a natural language and those of the algebraic language. A child acquires a natural language through an experimental attitude, gradually taking possession of the meanings and supporting rules of the language, which will then be developed at school age, through the learning of reading and reflection on the grammatical and syntactical aspects of language. Similarly, the mental models of algebraic thinking and language should already be constructed in an arithmetical environment, even from the very first years of primary school, teaching children to think of arithmetic in an algebraic way. In other words, this means progressively creating in the pupils' minds an algebraic thinking that is closely intertwined with arithmetic, exerting a continual reflection on the meanings of the introduced symbols and of the implemented processes.

The perspective to start off the students with algebra as a language, continually thinking back and forth between algebra and arithmetic, is based on the negotiation and then on the rendering explicit of a didactical contract, in order to find the solutions of problems, based on the principle "***first*** *represent,* ***then*** *solve*". This perspective seems very promising when facing one of the most important issues in the field of conceptual algebra: the transposition in terms of representation from the verbal language, in which problems are formulated or described, to the formal algebraic language, into which the relations contained in them are translated, and subsequently their solution. From this viewpoint, the translation of phrases from the natural (or graphic or iconic) language to the mathematical language and vice versa represents one of the most fertile areas within which it is possible to develop reflections on the mathematical language.

Such an innovative vision requires a process of authentic reconstruction of teachers' conceptions in the field of mathematics and methodology, and the definition of a new classroom culture, through the creation of an environment which might

---

1 The ArAl Project is led in collaboration with G. Navarra, a teacher researcher who co-ordinates the organizational aspects of the project and contributes to its scientific co-ordination.

informally stimulate the autonomous processing of *algebraic babbling*[2] i.e. the experimental and continually redefined mastering of a new language, in which the *rules* may find their place just as gradually, within a teaching situation which is tolerant of initial, syntactically “shaky” moments and open to reflection and comparison of activated representations and which stimulates a sensitive awareness of the formal aspects of the mathematical language.

The ArAl project units

The project’s scientific setup is illustrated in the “Theoretical Framework” and in the “Glossary” (http://www.matematica.unimo.it/0attività/Aralproject/ and Malara and Navarra 2003a). The Glossary, which was conceived with the aim of aiding teachers in their approach to theory, through the clarification of specific conceptual or linguistic constructs, constitutes in many aspects the *heart* of the ArAl Project. It currently includes 70 terms, all interconnected via a closed system of cross-references and hypertext links.

Nonetheless, the supporting pillars of the ArAl Project are the so-called “Units”, which are models of teaching processes of arithmetic in an algebraic perspective. Their preparation for publication takes place through a complex process, which lasts about three years. The units show the results of the progressive refinement of numerous experimentations and cross-analyses of diaries and/or records of classroom activities, of comparisons between and reflections of university researchers and teachers who are involved in the project in various ways, within the framework of a structured methodology (see below). It must be noted that all activities are introduced in the programs of each class and are carried out within normal working hours. Currently the project’s published units are eight in number, whilst three others are being developed.

The units are structured in such a way as to describe, in the left-hand column of each page, a reasoned sequence of resumés of didactical pathways followed, with their respective construction modalities, and so as to make transparent, in the right-hand column, those aspects that were derived from an analytical reading of the class diaries, which can help teachers in the application (methodological choices, activated classroom dynamics, key elements of the processes, extensions, potential pupils’ behaviour, difficulties that could be encountered, etc.). The final goal is to offer teachers the opportunity to reflect on their own knowledge and *modus operandi* in the classroom.

Situations concern explorations of: number grids, towers of numerical bricks, multicoloured necklaces containing various bead modules, equilibrium situations with two-plates scales, board games involving translation between natural and formal language, etc. Through these explorations, pupils approach the use of letters; activate the construction and processing of first algebraic expressions; construct and solve equations, reflecting on the underlying processes; interpret the meaning of formal writings in relation to specific issues. Pupils thus acquire an algebraic way of looking at arithmetic.

Situations are developed in teaching environments that are stimulating, but often not easily manageable, and they involve for the teachers numerous delicate aspects

---

[2] The theoretical construct of algebraic babbling has its roots in a vision of teaching-learning of algebra to be carried out in analogy with the teaching-learning of natural languages (see Malara, 1994). It is noteworthy that embryonal forms of such a construct can be found in the writings of Meira (1996).

requiring a corresponding number of competences. In the first place, they find themselves having to reckon with their own knowledge and convictions, accompanied also by a number of methodological and organizational aspects which are far from marginal and which operatively support a *culture of change*.

The project also becomes, for teachers, an important opportunity for reflecting on their own knowledge, both mathematical and methodological, and on their personal convictions (factors which in actual fact condition their choices) – indeed one could say *on their epistemology*.

Teachers and the ArAl project

On the experimental side, the project is supported by a network of schools in northern Italy, which involves 73 primary classes, 26 middle school classes and 2 nursery schools, with a total of 104 teachers and 2,485 pupils.

During the last few years, four different teacher roles have come to light within the project, based on the particular function: *teacher researcher* (TR); *teacher researcher in training* (TRT); *teacher experimenter* (TE); *teacher user* (TU).

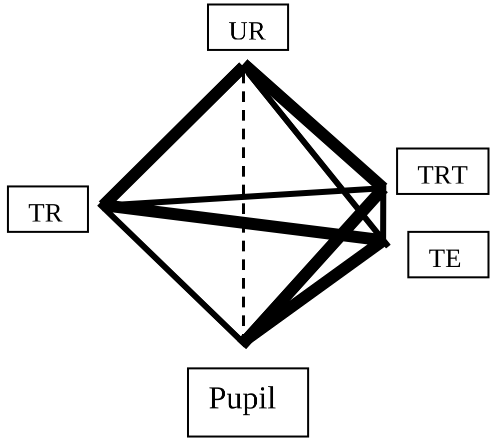
Teacher researchers are secondary school teachers, historically members of the Research Group (GREM) who have attained the actual status of researchers (i.e. they independently publish articles, lecture at conferences, and are also assigned co-ordinating and/or scientific revision roles). This teacher figure is a peculiarity of Italian research, being non-existent or rather weak in other European contexts, and represents an important link between the world of research and that of education (Malara and Zan, 2002). Within the project, these teachers contribute with specific themes to the definition of the theoretical framework, to the planning of materials and activities and their verification, to the analysis of processes, and act as models in joint-teacher class activities.

Teacher researchers in training are mainly from the lower grades of secondary schools. They are both serving teachers who have come into contact with GREM as winners of training scholarships for research within the framework of agreements between MIUR and Italian mathematics departments, as well as young and capable teachers, fresh from specialisation school and anxious to deepen their training on the theoretical and classroom observation side. In the last few years, both their number and their role have gradually become consolidated to the point of being, today, an important project component, for their new and vital way – beyond their mere involvement – in which they move both at the theoretical level (compilation of surveys, conducting seminars, etc.) and at the practical level (leading discussions, checking and reflecting on actuated processes).

Teacher experimenters belong mainly to primary schools and represent the most numerous component in the project. They work in joint classes alongside *teacher researchers*: they observe classes, contribute to the compilation of class diaries, to the analysis of pupil protocols, etc., contribute to the clarification and fine tuning of didactical pathways, also by sharing their reluctance, doubts or difficulties.

Teacher users represent a relatively recent component and are a consequence of the project's spreading on a nationwide scale. These are teachers from schools that make use of ArAl units and require a relationship of co-operation with GREM. They constitute a component for which the modalities for experimenting with e-learning processes (chat, forum, etc) are currently being studied.

The methodological structure



Given the contribution of these various components, the methodological structure of the ArAl project, though having its roots in the Italian model for innovation research, constitutes an important and complex evolution of it. This is its current model, with a background of new technologies, which significantly support the complex network of exchanges, launch the project into the surrounding world and open new scenarios for it.

The edges of the model in bold face highlight the privileged relations, namely: those between TR and TE, between university researcher (UR) and TR, and between UR and TRT, which are based on trust and dialogue. The TR plays a strong mediation role between UR and TE's, both on the theoretical and on the practical side. Experimental activities are started in primary school classes by the TR's, in joint classes with TE's. Class collaboration between the TE and TR encourages a hot confrontation in the face of emerging habits, stereotypes, convictions, misconceptions, etc., and encourages the TE to express points of view, doubts, perplexities, important indicators of his or her conceptions. In the case of the TRT, class activity is recorded and analysed in collaboration with the UR, who is directly responsible for this, and is then shared with other TRT's.

The joint analysis of classroom paths, both done in pairs, between TR-TE or UR-TRT, and in the whole group, reveals conceptual knots of the intertwining between arithmetic and algebra, and provides an opportunity for disclosing conceptualisation gaps in the mathematics education of the TE's. These gaps can then be the object of a critical analysis. The objective of this path is a dual one: on the one hand, it aims to lead the individual teachers to critically reflect on their work in the classroom, also through a comparison with the work of their colleagues; on the other hand, it aims to produce a comparative analysis of implementation in various classes of the units in progress, allowing for fine tuning of a prototype, which can then be widely circulated.

The collective work phase of comparative analysis of individual paths is of fundamental importance to teachers, since, by comparing their own path with that of their colleagues during the same steps of a didactical sequence, each of them can identify important differences and reflect on the efficacy or limitations of their own work. This leads them to a greater awareness of their own behaviour in the classroom, and to exert a tighter control of attitudes – and indeed to modify them (see Malara, 2003b).

This process also allows the emergence of teachers whose classroom behaviour is exemplary from the point of view of orchestration of discussions (restarts, silent pauses, spaces between words and self-control), and whose process protocols – enriched by significant personal comments – become a good study model for teachers in training (see Malara et al., 2004).

This methodology thus makes it possible to conduct activities on three distinct but concomitant levels (*research, experimentation, education*), tackling issues that are strongly intertwined between conceptions and personal attitudes on the one hand and didactical methodologies on the other.

Aspects of practice and highlighting of the classroom culture

We shall now analyse some classroom scenes of ArAl paths realised by a TRT[3] in a third-year primary class[4]. Our choice was motivated by the exemplary nature of this case, as regards both teacher's and pupils' behaviour and, more generally, for the implications of the educational aspects.

Our analysis will be concerned with the teacher's actions (didactical choices, her way to place herself in relation to the pupils, reflections, emotions), and with the classroom culture (atmosphere, discussions, individual pupil contributions, relationships between pairs or small groups, etc.). We shall also see, through the teacher's reflections, her own self-observation during the process, the impact of her emotions and convictions on her didactical choices, the establishment of new levels of awareness of the potential of discussion for the collective creation of meanings, and the importance of the metacognitive dimension within the teaching-learning process.

A third-year class path on the unit "From scales to equations"

The unit, conceived for 10/11-year-old pupils, is structured from experience to theory and uses a pair of scales – in actual fact virtual ones – to bring about, through a representation process of the examined equilibrium situations, the birth of the equation and the objectivation of the underlying principles of its solution.

After an introductory phase working with the scales, problems are then set in diversified contexts and are of increasing difficulty, ending up being rather complex.

The teacher tackles this experimentation with concerned uncertainty, since the unit is geared to older pupils, and she is worried that division, being an operation which at a certain point gets in play, may not yet be sufficiently well grasped by the class.[5] The activity, planned by the teacher, is completed in a total of 13 hours, divided into: *preliminary phase, action phase, representation phase, conclusion phase*.

The preliminary phase is devoted to a discussion of equilibrium, aiming to assess the pupils' conceptions and check whether they already grasp the scales as a model of equilibrium. The aim of this phase is the objectivation of the general principle of the scales, namely that "the scales are in equilibrium if – and only if – the weights lying of their pans are equal".

The action phase concerns the collective exploration, with the virtual scales, of equilibrium situations (the so-called "Wizard's Riddles"), until a solution is found (4-5 hours). The aim of this phase is the objectivation as "theorems in action" (as intended by Vergnaud) of the two "principles of the scales": 1st principle: "If the scales are in equilibrium, removing equal weights from their pans, the scales remain in

3 The teacher's name is Elisabetta Magnani.

4 The class, made up of 22 pupils, is a middle-level one, including 5 high-level children, 5 at middle-to-low level, and 3 problem cases.

5 Inserted into a group with other colleagues engaged in the Scales Unit, attracted by her colleagues' achievements and "to avoid feeling isolated", the teacher decides to tackle the unit's initial part, also encouraged by the results she has already obtained in the class working on another ArAl unit.

equilibrium"; 2nd principle: "With scales in equilibrium, if the weights on the pans are divided by the same number, the scales remain in equilibrium".

The representation phase is a complex one, and consists of specific activities geared towards the representation of: a) scales and equilibrium; b) unknown data; d) processes for the solution of the Wizard's Riddles.

The objectives of the various stages of activity are: a) the progressive simplification of representations of the scales' equilibrium, in search of a shared symbol to represent the equality of the weights lying on the pans; b) the introduction of symbols to indicate unknown quantities; c) the approach to equations, seen as an equality of different representations of same quantities and their solution through the codification of completed actions and of the principles that were implemented for their solution.

The conclusion phase is a checking phase of the pupils' learning and, more generally, a phase of wider reflection on what has been done. More specifically, it concerns the creation of riddles in pairs and their representation, on the model of the ones previously examined; also, the pupils answer a self-assessment questionnaire.

The activities of the different phases are separated by, on average, a 5-day gap.

We shall now dwell on some "scenes" from this path[6].

On the representation phase

Of the various moments of this phase, though all of them very interesting, we wish to dwell on a significant episode concerning the metacognitive control of the teacher's own actions as a the teacher. During the action phase in the collective discussion the children had tackled the solution of the following riddles:

*1st Riddle: We have on the scales: left-hand pan, a bag of flour and 50 g; right-hand pan, an 80 g weight***.** How can we find out how much a bag of flour weighs?

2nd Riddle: We have on the scales: left-hand pan, 3 weights, 2 weighing 30 g each and 1 weighing 20 g; right-hand pan, a packet of little pasta stars and 2 weights, 1 weighing 30 g and 1 weighing 20 g. *How can we find out how much the packet of little stars weighs?*

3rd Riddle: We have on the scales: left-hand pan, 2 identical packets of salt; right-hand pan, 200 g. *How can we find out how much a single packet of salts weighs?*

4th Riddle: We have on the scales: left-hand pan, 3 identical packets of salt; right-hand pan, 300 g. *How can we find out how much a single packet of salt weighs?*

During the representation phase the children had been involved in the following activities.

1. They had tackled collectively the representation of the equilibrium of the scales and had chosen, in a vote that followed the discussion, this symbol, ————— seen by the children as being the most representative of equilibrium.
2. They had faced the representation of the first riddle, with the instructions: "Try to

6 A detailed analysis of the didactical processes from which the excerpts are taken appears in Malara et al., 2004.

tell what we did to solve the Wizard's Riddle with the language of mathematics, which uses numbers, signs, operation symbols, letters" and introduced, without any serious problems[7], the letter to indicate the contents of a packet, even though with little awareness of its meaning (most children see it as an abbreviation of the name). Some of the representations produced by the children are reported below.

Luca *F + 50g* | *80g* *F + 50 = 80* ~~*F + 50*~~ | ~~*50 +*~~ 30 *30* *F = 30g*	Laura *F. 50g = 80g* F 50 = 80 F 50 = 50 F 50 = 50 30 30 =
Giommi *F. 50g = 80g* *50g + F = 80g* *50 + 30g = 80g*	Greta B *F. 50g = 80g* 80 - 50 = 30

Following this first test, the teacher is *convinced that the children need help in structuring the representation*, and discusses again with them the representation of the 1st riddle, especially its central phase. This is the discussion:

> *Teacher*: Children, the right steps have all arisen, now we want to put them into the right sequence. Many of you have written the right beginning, the initial state of balance (she writes the representation on the blackboard).
>
> *F 50g = 80g* *beginning*
>
> Then you said that at the end the flour weighted 30 g. How can I write this with the language of mathematics? (*The final equality doesn't come from the pupils*)
> *Teacher* (again): How did we write about salt in the previous riddle? What happens?
> *Giorgia*: F is equal to 30g
> *Teacher*: Good, let's record at the bottom: F = 30g; the end. What about the middle part? Many have written: F + 50 = 80, but I remember taking 80 away and putting…

[7] The children had already met letters, working on a previous ArAl unit.

> *Luca*: 50 and 30
> *Teacher*: Right, so we have F + 50 = 50 + 30. At this point we need Marghe's sign, she had some crosses there
> *Giommi*: Let's take 50 and 50 away
> *Teacher*: How can we do it?
> *Lara*: With a slash
> *Teacher*: Ok, this is the first principle: F + ~~50~~ = ~~50~~ + 30, that's why we need to see that we take away from both sides
> *Chiara B*: Because even if you take away, the scales are in balance".
> (The whole procedure is recordes in the arranged form.)

In spite of the discussion's good results and the participation of the class, the teacher's reflection is very interesting:

> "I realise that if I continue to conduct the representation collectively, I'll try with all means to lead the children to a writing that is "correct for me as a teacher", without taking into account their difficulties and logic and/or comprehension impediments, and without understanding who could actually reach a solution, though perhaps different from my own. I've decided that, from now on, I will help the children, but I will let them operate individually, also to see their actual degree of involvement in the activity."

We regard this excerpt as very indicative of the calibre of the teacher, who not only manages to exert a constant control of her own behaviour in action, but also places herself on a level of "aims reflection", examining her own actions in the context of the whole class process, and reflecting on the impact of these actions on the children's behaviour and on the effects of their learning.

Just as examples, we show in the following table some riddles, with their corresponding representations, which the pupils "invented" for the Wizard. [Table p. 12.]

Giommi and *Davide*	On the left-hand pan there are a packet of salt and a bag of flour. On the right-hand pan there are a packet of salt and a 300 g piece. How much does the bag of flower weigh?	**Fundamental** **Principle** 1st Principle	*Giommi* F = S 300g S + F = S + 300g F = 300g
Greta C, *Laura* and *Chiara*	On the left-hand pan there are a packet of little pasta stars and a 50 g weight. On the right-hand pan there is a 90g weight. How much does the packet of pasta stars weigh?	**Fundamental** **Principle** 1st Principle	*Greta C* A 50g = 90g 90g – 50g = 40g A *Laura* A 50g = 90g A 50g = 50g 40g A = 40g
Giulio and *Renato*	On the right-hand pan there are 3 small packets of salt. On the left-hand pan there is a 600 g weight. How much is a small packet of salt worth?	Fundamental Principle 2nd Principle	*Giulio* O.O.O. = 600 200+200+200=200+200+200 *Renato* 600 = SSS = 200 200 200 200+200+200=200+200+200 600 : 3 = 200

Note, in Giulio and Renato's protocol, the difficulty which the pupils encounter in representing the solution process. There are several reasons for this, among them probably a lack of conceptualisation of division and of its associated operators, as well as the absence of adequate representation tools, but there is also the need to invent a means for overcoming, in the representation, the conflict between action (dynamic) and representation (static) and the objectivation – through appropriate signs – of the principles used.

The metacognitive dimension

We conclude our observation with the analysis of a self-assessment questionnaire handed out to the children at the end of this experiment.

1. The children assess what they have done and learnt

The questionnaire:

WHAT TITLE WOULD YOU GIVE TO THIS ACTIVITY?

- DID YOU LIKE IT?

- WHICH PART DID YOU ENJOY MOST?
- WHAT WAS MOST DIFFICULT?
- WHAT DID LEARN? Here are some possible answers, but you may add others and choose the ones that are right for you:
    - NOTHING
    - TO COUNT
    - TO REFLECT
    - STRANGE THINGS, for example:
    - TO TALK
    - TO DISCUSS
    - TO LISTEN TO OTHERS
    - TO EXPRESS MY THOUGHTS
    - TO CALCULATE WITH NUMBERS AND LETTERS
    - OTHERS…

We focus our attention on the answers relating to learning. This was their distribution: I learnt: To reflect - 11; To talk – 3; To discuss - 8; To listen to others - 9; To express my thoughts - 11; To calculate with numbers and letters - 11; Strange things – 5; Others - 7. The answers given to the open question "I LEARNT STRANGE THINGS" concerned: thinking; expressing one's own thoughts with numbers; application and concentration; thinking with one's own head; deciding together with others; thinking collectively.The teacher writes in her comment:

> Positive aspects of the whole activity which clearly emerge are "Reflecting", "Expressing mathematical thought" and "Using numbers and letters". This seems to me to be profitable and useful, both for the teaching of mathematics and for any other activity**.**

2. The teacher assesses the experience, both for the class and for herself
We show here some excerpts of the teacher's final report.

> *"1. Impact of the experience on the class*
> The experience, which was assessed by the children as decidedly positive, generated curiosity, interest, and, on the whole, a good degree of participation. The following could be some of the factors that had a positive influence: a) The diversity, compared with the daily routine, and the almost exclusively "oral" work, compared with the prevailing writing work, found in mathematics (so many columns operations!); b) The atmosphere of the work context, which was unique, involving, with all the children in a circle around a board, or at the blackboard, or at the scales: a bit of magic, where the Numbers, Wizard launches challenges, which are accepted by the children. There are prizes, at first sweets, then a tag bearing the phrase "You're GREAT!", and, finally, the personal satisfaction derived from the solution of the problem. The activity produced an effect at several levels.
>
> *a) Relational-social level*. The activity allowed all children to be involved, particularly those belonging to the middle and lower groups, who normally only tag along. There has been an activation of metacognitive and transversal capacities that are fundamental for communication and learning: attention, listening, reflection, expression in natural language of the children's own opinions, acceptance of

criticism, collective decisions and even renunciation of personal positions.

b) Disciplinary level. The activity allowed the strengthening of some fundamental concepts of arithmetic: the number and its representation; operations linked to one another (simple expressions); equality; the letter as the "unknown entity" or as a "generic number". It has been possible to examine in greater depth and to strengthen the mathematical language, both natural and symbolic, in particular that of multiplication notation. I remember our discussion on the terms "double", "do twice", "multiply" (we saw the impact of the different meaning of the two terms in multiplication as seen as repetitive addition and determined the importance of commutativity in putting them both on the same level).

c) Difficulties level. There emerged clear linguistic difficulties: for an eight-year-old child, it is still a great achievement to be able to correctly express thoughts, both as regards the semantics, given the lack of clear meanings (e.g. that of "division"), and the syntax, given the poor logical structuring of thought and phrase.

2. The impact of this experience on me as a teacher

The experimentation we carried out was extremely positive and strongly motivating from a professional viewpoint. It produced in me relapses and multiple effects. On the methodological level, the most important aspect concerned the strengthening of my capacity to: make decisions and micro decisions; be able to accept and tackle any deviation from a pre-ordained path; conduct a "real discussion" in the classroom, and analyse its components, observing the importance of my silent pauses, the relevance of the children's contributions and silent moments; the preparatory and forecasting revision of the conversation; the conclusion shared by all, including the teacher (who would sometimes prefer different answers).

The recordings of the lessons made it possible for me to have a continuous feedback on my operating modalities, sometimes high-lighting procedural mistakes, with missed or excessively prevaricating interventions, doubts about mathematical concepts, hesitations… .

On the theoretical-disciplinary level, the confrontation with teachers who are mathematically competent has allowed me to look in greater depth at various arithmetical and algebraic themes, the latter being for me the more difficult ones".

Some considerations in closing

The complexity of the ArAl project entails an equal degree of complexity in the observation and analysis of the didactical phenomena taking place within it. Numerous are the variables to be taken into consideration (pupils, teachers, researchers), as also many are the viewpoints from which they can be examined in their reciprocal relationships: the pupils (in the class dynamics, in their individual productions, in learning); the teachers (in the classroom, with their ArAl project colleagues, with their researcher contacts); the researchers (what they would like to achieve, what actually happens, the new research problems they pose themselves, etc.). Against the background of all this, new project units are being created, through the analysis of the single experimentations and the comparative analysis of parallel experimentations.

Here, we concentrated our attention on the teachers and their role in the development of a classroom culture in a pre-algebraic key. Through the study of one case, we have tried to show a cross-section of the intricate intertwining of operativity, study, reflection and comparison, which underlies the work carried out in the framework of our project. It must be stressed that this is not an isolated case, but is actually typical of our operative standards (for an analysis of other cases, see Malara 2003a, 2003b, Malara et al. 2004).

Of course, not all teachers display the same qualities as the one we observed here, of exemplary competence, her great attention to the involvement of the entire class, her awareness of what she is doing and her constant reflection on what is happening. In this regard, we should refer back to some of the considerations from our plenary at PME 27 (see Malara, 2003a). Often, in the midst of live classroom action, teachers do not grasp a pupil's reasoning, or fail to give it due recognition, thus allowing significant contributions to be dropped, or they are conditioned by some pupils' invasiveness, or even unable to use appropriate silent pauses. All this very clearly shows us the importance of a fine teacher education as regards reflection – both local, in relation to their action on the spot, and global, on the modalities and sense of their operating, so that they may rethink those occasions of missed interventions, of inappropriate decisions, and of how they might have been different and better.

An effective strategy for the growth of teachers in this respect appears to be the cross- comparison of parallel interventions. By comparing their own path with that of other colleagues along the same steps of an identical didactical sequence, teachers can reveal important elements of difference and reflect on the productivity or limitations of their work. This offers an opportunity to reflect on disclosed bad habits, on the underlying conceptions or emotions, leading them to the acquisition of a greater awareness of their way of being in the classroom, and to exert a greater degree of self-control of their attitudes and even to slowly modify them.

Globally speaking, after observation of the "actors ensemble" involved in the project, we can confirm that in general:

As regards the teachers:

- An evolution of knowledge and a maturation of a new awareness, especially as concerns a critical re-reading of their basic knowledge, acquired almost exclusively by attendance at a (nearly always traditionally structured) higher education establishment or a non-mathematical university faculty;
- The refining of the sensibility in catching – at the very act of their arising – the potential of the pupils' contributions, their intuitions, and even the obstacles created by distorted or partial visions, very often extremely difficult to recognise, especially in pupils of this age.

As regards the pupils:

- The acquisition, through forms of algebraic babbling, of a *friendly* attitude towards the use of letters for codifying and generalizing observed facts;
- More generally, the achievement of a vision and appreciation of mathematics as a constructive discipline.

As regards us, the researchers:

- An aspect to be examined in more depth and anyway a *strong* element for a critical reading of the results concerns the possibility of observing in a more consistent way, and therefore more significantly – at least potentially – the component of *continuity between primary and secondary school*, in order to assess the relapses into an anticipated approach to algebraic thinking – or, in other words, of an approach to arithmetic in an algebraic key – on the evolution of the pupils' mathematical thinking.

References

Ainley, J.: 1999, Doing algebra type stuff: emergent algebra in the primary school, *Proceedings of PME 23*, vol. 2, 9-16

Ainley J. (coord.): 2001, Research Forum on Early Algebra, *Proceedings of PME 25*, vol. 1, 128-159

Arcavi A.: 1994, Symbol sense: informal sense-making in formal mathematics, *For the Learning of Mathematics*, **14**, 3, 24-35

Arzarello, F., Bazzini, L., Chiappini, G.: 1993 Cognitive Processes in Algebraic Thinking: Towards a Theoretical Framework, *Proceedings PME 17*, vol.1, 138-145

Bell, A.W., Onslow, B., Pratt, K., Purdy, D., & Swan, M.B.: 1985, *Diagnostic Teaching: Teaching from long term learning*, Report of ESRC project 8491/1, Schell Centre for Mathematical Education, University of Nottingham

Blanton, M., Kaput, J.: 2001, Algebrafying the elementary mathematics experience: transforming task structures: transforming practice on a district-wide scale, in Chick, E. et al. (Eds), *'The future of the teaching and learning of Algebra', Proceedings of 12th ICMI Study*, vol. 1, 87-95

Blanton, M., Kaput, J.: 2002, Design principles for tasks that support algebraic thinking in elementary school classroom, *Proceedings of PME 26*, vol. 2, 105-112

Both, L.: 1989, A question of Structure, in Wagner, S., Kieran, K (Eds.), Research issues in the *learning and teaching of mathematics*, NCTM, VA: Lawrence Erlbaum Associates, US, 57-59

Brown, L., Coles, A., 1999, Needing to use algebras – A case study, *Proceedings PME 23*, vol. 2, 153-160

Brito Lima, A.P., Da Rocha Falcão, J.T.: 1997, Early development of algebraic representation among 6-13 year-old children: the importance of didactic contract, *Proceedings of PME 21*, vol. 2, 201-208

Boulton Lewis, G.M., Cooper, T, Atweh, B., Pillay, H., Wilss, L.: 1998, Pre-algebra: a cognitive perspective, *Proceedings of PME 22*, vol. 2, 144-151

Carraher, D., Brizuela, B., Schliemann, A.: 2000, Bringing out the algebraic character of Arithmetic: instantiating variables in addition and subtraction, *Proceedings of PME 24*, vol. 2, 145-152

Carraher, D.: 2001, Can young students operate on unknowns? *Proceedings of PME 25*, vol. 1, 130-140

Chick, E., Stacey, K, Vincent, Jl., Vincent, Jn. (Eds.): 2001, *'The future of the teaching and learning of algebra'*, *Proceedings of 12th ICMI Study*, University Melbourne, Australia

Da Roca Falcão, J.T.: 1995, A case study of Algebraic Scaffolding: from balance to algebraic notation, *Proceedings of PME 19*, Recife, vol. 2, 66-73

Da Rocha Falcao, J.T.: 1996, Clinical analysis of difficulties in algebraic problem solving among Brazilian students: principal aspects and didactic issues, *Proceedings of PME 20*, vol. 2, 257-263.

Da Rocha Falcão, J.T., Brito Lima, A.P., De Araújo, C. R., Lins Lessa, M.M., Osório, M.O.: 2000, A didactic sequence for the introduction of algebraic activity in early elementary school, *Proceedings of PME 24*, vol. 2, 209-217

Dauguerty, B.: 2001, Access to algrebra: a process approach, in Chick, E. et al. (Eds.), *'The future of the teaching and learning of Algebra'*, *Proceedings of 12th ICMI Study*, vol. 1, 344-353, 87-95

Filloy, E.: 1990, PME algebra research. A working perspective, *Proceedings of PME 14,* vol.1, 1-33

Gray, E., Tall, D.: 1993, Success and failure in Mathematics: the flexible meaning of symbols as process and concept, *Mathematics Teaching,* **142**, March, 6-10

Harper, E. (Ed): 1987-88, *NMP Mathematics for Secondary School*, Longman Essex, UK

Herscovics, N.: 1989, Cognitive obstacles encountered in the learning of algebra, in Wagner,S., Kieran, K. (Eds.), *Research issues in the learning and teaching of mathematics*, LEA, NCTM, Virginia, US, 60-86

Herscovics, N., Linchevski, L.: 1994, A cognitive gap between Arithmetic and Algebra, *Educational Studies in Mathematics*, **27,** 59-78

Kaput, J.: 1991, Notations and representations as mediators of constructive processes, in von Glasersfeld, E. (Ed.) *Radical Constructivism in Mathematics Education*, Kluver Academic Publishers, The Netherlands, 53-74

Kaput J., Blanton M.: 2001, Algebrafying the elementary mathematics experience: transforming task structures, in Chick, E. et al. (Eds.), *'The future of the teaching and learning of Algebra*', *Proceedings of 12th ICMI Study,* vol. 1, 344-353

Kieran, K.: 1992, The learning and teaching of school algebra, in Grouws D.A. (Ed.), *Handbook of research on mathematics teaching and learning*, Macmillan, NY, 390-419

Lins, R.C.: 1990, A framework for understanding what Algebraic Thinking is, *PME XIV*, vol. 2, 93-100.

Linchevski, L.: 1995, Algebra with numbers and arithmetic with letters: a definition of pre-algebra, *Journal of Mathematical Behaviour*, **14**, 113-120

Malara, N. A.: 1994, Il pensiero algebrico: come è possibile promuoverlo nella scuola dell'obbligo limitandone le difficoltà? in D'Amore, B. (Ed.) *Dalla ricerca teorica alla pratica didattica,* Pitagora, Bologna, 67-77

Malara, N.A.: 2003a, Dialectics between theory and practice: theoretical issues and aspects of practice from an early algebra project, plenary lecture, *Proceedings of PME 27,* vol. 1, 33-48

Malara, N.A.: 2003b, Promoting teachers' changes: examples from an educational process in early algebra, plenary lecture, in A. Rogerson (Ed.) *Proceedings of International Conference 'The Mathematics Education into the 21st Century'*, University of Brno, Repubblica Ceka, 5-16

Malara, N.A, Iaderosa, R.: 1999, Theory and practice: a case of fruitful relationship for the renewal of the teaching and learning of algebra, plenary lecture, *Proceedings of CIEAEM 50,* 38-54

Malara, N.A., Navarra, G.: 2000, Explorative ways to encourage algebraic thinking through problems, in Gagatsis, E, Makrides, G. (Eds.) *Proceedings of II Mediterranean Conference on Mathematics Education,* vol. 1, 55-64, also in *L'Educazione Matematica*, XXVI, VI, n.1, vol. 2, 7-21

Malara, N.A., Navarra, G.: 2001, "Brioshi" and other mediation tools employed in a teaching of arithmetic with the aim of approaching algebra as a language, in Chick, E. & al. (Eds.), *'The future of the teaching and learning of Algebra' Proceedings of 12th ICMI Study on Algebra*, vol. 2, 412-419

Malara, N.A., Navarra, G.: 2003a, *ArAl project: arithmetic pathways towards favouring pre-algebraic thinking*, Pitagora, Bologna

Malara, N.A., Navarra, G.: 2003b, Influences of a procedural vision of arithmetic in algebra learning, WG 2 - CERME 3, (Bellaria, Italy, february 2003)

Malara, N.A, Zan, R.: 2002, The problematic relationship between theory and practice, in English, L. (Ed.) *Handbook of International Research in Mathematics Education,* LEA, NJ, 553-580

Malara, N.A., Incerti, V., Fiorini, R., Nasi, R.: 2004, *Percorsi di insegnamento in chiave pre-algebrica: rappresentazione di problemi e di processi, segni simboli e negoziazione dei loro significati*, Pitagora, Bologna

Meira, L.L.: 1990, Developing knowledge of functions through manipulation of a physical device, *Proceedings of PME 14,* vol.2, 101-108.

Meira, L.: 1996, Students'early algebraic activity: sense making and production of meanings in mathematics, *Proceedings of PME 20.* vol.3, 377-384

Menzel, B.: 2001, Language conceptions of algebra are idiosincratic, in Chick, H. et al. (Eds.) '*The future of the teaching and learning of Algebra*', *Proceedings of 12th ICMI Study*, vol. 2, 446-453

Pegg, J., Redden, E.: 1990, Procedures for, and experiences in, introducing algebra in New South Wales, *Mathematics Teacher*, **83**, 5, 386-391

Radford L.: 2000, Signs and meanings in students' emergent algebraic thinking: a semiotic analysis, *Educational Studies in Mathematics*, **42**, 3, 237-268

Radford L., Grenier M.: 1996, On dialectical relationship between signs and algebraic ideas, *Proceedings of PME 20*, vol. 4, 179-186.

Wagner,S., Kieran, K.: 1989, *Research issues in the learning and teaching of mathematics*, LEA, NCTM. Lawrence Erlbaum Associates, Virginia, USA

Usiskin, Z.: 1987, Why elementary algebra can, should, and must be an eighth-grade course for average students, *Mathematics Teacher*, **80**. 9, 428-438