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Proof, proving, and the work of teachers and students in classrooms

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Abstract

This paper provides broad strokes of a program of systemic research on the place that proof occupies in high school geometry instruction in the United States. The historical emergence of activities where student produce proofs is described. Then the main elements are outlined of a descriptive model of the instructional situations in which students do proofs in contemporary geometry classes. Questions are posed and an emerging program of research is described that addresses the practical rationality that teachers invest in creating and managing those instructional situations.

Introduction

Proof and proving have been objects of research for mathematics educators for quite a while. Indeed, a relatively large group of researchers have been interested in the teaching and learning of mathematical proof as well as in the development of mathematical reasoning in individuals of all age groups. Some of these researchers have started from an understanding of what mathematical proof is in the discipline of mathematics. Other researchers have started from an understanding of the mechanisms by which individuals develop their capacities to reason. Much research has been concerned with questions such as “How do students learn to do proofs?”, or “How can proof be taught?,” or “What are the cognitive structures that allow or prevent individuals to reason deductively and produce proofs?,” or even “How does skill in producing proofs develop?” Some of those questions have the flavor of basic psychological research, attempting to describe and explain phenomena of individual cognition; others have the flavor of applied educational research attempting to understand or evaluate the qualities of educational treatments in terms of their effects on individual learning.

In spite of the diversity that one can find in such research on proof, a common characteristic of these approaches has been that they (implicitly) conceive of classroom instruction as a means-to-an-end. There is something commonsensical about that. Isn’t classroom instruction a process put in place to facilitate the main educational goal of producing student learning? Indeed; especially so from the perspective of the (Western) institution of schooling: Schools and classrooms justify their existence on instrumental grounds - be those grounds the development of people, or the reproduction of knowledge. Much research in education is actually “house” research, research *inside* education - research that sooner or later informs the institutional agenda of schools, their publicly avowed purposes and their derived policies. It is important to ask if there is any room for research beyond those boundaries. And what does this apparently general discussion have to do with proof and proving? Without dismissing the former, I intend to make the case for one (though

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not necessarily *the* one) different kind of research on mathematics classroom instruction, to argue for the centrality of a *problématique* of proof and proving in that kind of research, and to provide some examples from my own work in that *problématique*.

In the last twenty years, claims have been staked on issues relevant to proof and proving from a different perspective, one that looks at classroom instruction as a self-regulating system operating under conditions and constraints (some of which are the need to achieve certain publicly avowed ends) and producing, as a result of such operation, a variety of cultural products. Some of those cultural products are of interest to the sociologist who may ask whether and why classroom instruction reproduces, at the micro level, distributions of social capital existing in the larger society. For the mathematically oriented observer, analogously, mathematics instruction produces interesting (and oftentimes oddly interesting) kinds of mathematical activity as it attempts to accomplish its mission within its conditions and constraints.

To make this different perspective on classroom instruction more apparent let me suggest a parallel with film. Film can also be seen as a means-to-an-end, whether one thinks of that end as public entertainment, studio's profit, political propaganda, or actor's skill demonstration. Yet the discipline of film criticism is predicated on the possibility to examine the qualities of a film not in terms of how well it achieves any of those avowed ends but on how it represents the cultural objects that it handles. A film creates a representation of a cultural object (e.g., of family life, memory, time, work, etc.) whose singularity (or uniqueness) is partly a result of the need to operate its own dynamic in order to achieve those avowed ends (see Thomson, 2004). Classroom instruction is amenable to a kind of investigation similar to that which film criticism does of film: an investigation that takes instruction as an activity that produces unique ecologies of meaning at least partly in response to the particular conditions in which that activity takes place.

An observer of classroom work may attest to the existence, for example, of singular conceptions of a given mathematical idea (e.g., function or rational number) at play in particular tasks that students do. Some of those conceptions could be at odds with the conception of that idea prescribed by the curriculum, and yet still constitute a viable adaptation of the class to the specific demands of the task. A systemic approach to understanding classroom instruction would take the existence of those alternative conceptions not necessarily as an indication that something has been done wrong but rather as a more or less legitimate product of a system whose specific work trying to fulfill a variety of demands needs to be understood. One of the conditions in which many instructional systems exist is that teachers are accountable for their students' learning of content. Whereas an instrumental view on instruction would take students' learning as a goal to be achieved and would consider instruction merely as the process that it takes to achieve it, a systemic view would take this goal as a systemic constraint (or a demand) on instruction, and attend to what becomes available to be learned, as a result of instruction having to operate dynamically and respond to that accountability demand among others.

Such an approach to the study of instruction can be useful: In the same way that films are artifacts that represent and preserve cultural ideas and concerns of their time, mathematical activity in school classrooms is a reservoir of artifacts (memorable episodes perhaps) that represent and preserve Mathematics for individuals, for generations, and for societies. To the extent that these representations constitute most students' opportunities to learn, this ecological approach should have some appeal to those who are interested in knowing *what* may be learned. In that sense, the cultural meanings of proof and proving that are constructed in and through the work of instruction play a key part in constituting most people's images of the discipline of mathematics and of the nature of mathematical work.

The word "instruction" has many uses in ordinary parlance. I use it here to designate the system of practices that relate those who play the role of teacher and students in the mathematics class, specifically those practices that relate them vis-à-vis mathematics as a subject of their transactions (communication, study, etc.; see Chevallard, 1991; Cohen, Raudenbush, and Ball, 2003). The postulate that classroom mathematics instruction produces ecologies of meaning in response to the particular conditions in which instruction takes place locates mathematics in potentially two places. Mathematics is, on the one hand, what is officially at stake in that system of practices - what students are supposed to learn and the teacher is supposed to teach, what they acknowledge having taught or learned. But mathematics can also appear, on the other hand, as the set of meanings (that an observer can see) being handled in and through the play of that system of practices, embodied in classroom action. This latter sense in which mathematics appears in instruction deserves further comment: As in the observation of individuals' intelligent action, in the observation of larger organizations such as a class or a company, all that an observer can do is to construct more or less viable models of what is the mathematics (or the knowledge) that they seem to be handling. Yet it is clear that the mathematics that an observer can see being handled in and through the work of a class may or may not be compatible with the mathematics officially at stake in that work. As Brousseau (1997) shows when he identifies the effects Topaze or Jourdain, it is quite possible that one and the other mathematics may not be the same. The actors (teacher and student) may stake a claim on some mathematical object but an observer may make a mathematical reading of their actions that identifies a different mathematical object being handled - or, in extreme cases, none at all. Awareness of the possibility and the relevance of treating classroom instruction in a systemic rather than in an instrumental way has been raised by work done at both sides of the Atlantic, specifically by much of the work done in *didactique* of mathematics in France (e.g., Balacheff, 1999; Brousseau, 1997; Chevallard, 1991) and by the research on mathematics teaching done by teacher-researchers in the United States (Ball, 1993; Chazan, 2000; Lampert, 1990, 2001).

The foregoing discussion should anticipate the possibility to consider two aspects in which proof and proving are related to classroom instruction. First, proof and proving constitute (or are related to) a number of potential conditions and constraints on instruction. For example, certain instructional systems are charged with the responsibility to enculturate students in the practice of proving propositions. Second, proof and proving constitute tools that an observer may use to describe the nature of the mathematical rationality at work in a classroom. For example, proving might (or might not) describe the ways in which solutions to problems are found or the truth of claims is justified. Furthermore, the foregoing discussion suggests as an interesting pursuit to examine the relationship between one and the other manifestation of proof in such instructional system. Specifically, what place do proof and proving occupy in the mathematical work done by teachers and students in classrooms whose demands include the need to teach and learn mathematical proof? This has been the guiding question of my work in the past five years and a particularly important beachhead to develop means to think about what mathematics can be known and done in classrooms.

The place of proof in mathematics

From Lakatos's (1976) study of the historical development of Euler's conjecture concerning the relationship among the number of edges, vertices and faces of a polyhedron we learn that in the discipline of mathematics, proof and refutations play a key methodological role in the development - not just the justification - of knowledge. We can derive at least two key lessons about the place of proof in mathematics. One key lesson is that proofs do not just articulate well-defined concepts. Rather, the dialectic of proofs and refutations pushes forward a simultaneous uncovering and delimiting of the meaning of mathematical ideas, which Lakatos illustrates for the case of “polyhedron.” A second key lesson from Lakatos is that proofs do not just verify that a clearly stated proposition is true, they don’t just explain why it is true. Rather, while trying to do that, proofs and refutations actually push forward a simultaneous formulation of the proposition that can be asserted and calibration of the means by which it can be warranted.

Inspired by noticing the place that proof plays in the development of mathematical knowledge in the particular domain of the discipline studied by Lakatos, one could go into other mathematical settings, such as classrooms, and describe the place of proof. In such inquiry, Lakatos does not provide us with a descriptive assertion regarding what role proof actually plays in classrooms; nor does he provide a normative assertion regarding what role proof should play in classrooms. Instead, Lakatos’s work provides us with an icon or an image of what authentic mathematical activity across history looks like.[2] This icon can be instrumental for an observer as the backdrop against which to observe the singular place of proof in classrooms. If anything, that icon invites us to construct a sense of intellectual surprise when we face classroom work (or conversely, to deconstruct our sense of familiarity with ordinary classroom work, possibly developed by our having been teachers and students) and pose some questions:

> *How different is the place of proof in the development of mathematics in classrooms from the place proof has in the development of mathematics as a discipline? How are these differences related to the conditions in which classroom instruction operates?*

A stable place for proof in high school geometry in the United States

My own empirical work fleshing out this program of research has been concentrated on the high school geometry class in the United States. The interdependence of Euclidean geometry and proof in the US has a long history that probably resembles (at least in its inception) the history in other countries. Geometry used to be the reasoned discourse whose acquisition would educate students’ capacity to reason as well as transmit one of the main cultural achievements of the classical world. In the United States a course in geometry was originally a main staple of colleges and universities but became a high school course at some point in mid-19th century. Whereas high schools had originally been conceived as preparatory to college, near the end of the 19th century they took on the added task of socializing young immigrants to American life (Kliebard, 1986). These diverse purposes featured heavily in the debates at the end of the century on the content of the high school curriculum, especially on the purpose of the geometry course and the place of proof.

The curriculum recommendations of the so-called Committee of Ten (Eliot et al., 1893) near the end of the 19th century secured a place for the study of geometry for every high school student. Like in all other courses that found a place in the

[2] Clearly the goal is not to attribute any special prerogative or absolute truth to Lakatos’ account but to find a systematic way to distance ourselves from the “master narratives” of the “tribes” that designate classrooms as means-to-an-end (whatever that end is).

curriculum, the main argument made by the Committee was not the need to communicate to a new generation valued knowledge from various scientific and artistic fields but rather the need to provide discipline to the mental faculties. In particular, the Committee of Ten identified the high school geometry course as a vehicle for students to acquire the *art of demonstration* (or proving), since "geometrical demonstration is to be chiefly prized [as] a discipline in complete, exact, and logical statement" (Eliot et al., 1893/1969, p. 25). Unlike before, when students would prove the theorems of Euclidean geometry as a way of studying the domain of geometry (and as a byproduct of that study, possibly develop the art and habits of mathematical reasoning), the geometry course would now be predicated on the need to develop such art. In Herbst (2002) I have documented in detail the evolution undergone by the geometry course during the late 19th and beginning 20th century as a result of the need to ensure its success at having students acquire the art of demonstration. During the 20th century other arguments have been advanced for the need to study geometry in high school (Gonzalez and Herbst, 2004), among them the *formal* argument that geometry can be taught in such a way that students can develop their capacities for deductive reasoning to use in other aspects of life (Fawcett, 1938), and the *mathematical* argument that geometry provides an authentic encounter with the activity of proving theorems (Henderson, 1947). Both of those modal arguments have continued to stress the place of proof in the geometry course. Toward the end of the 20th century the American high school curriculum continued to be, for the majority of students, divided into separate rather than integrated domains of mathematics (students study Algebra, Geometry, Pre-Calculus, and Calculus in high school). And among those offerings, only the high school geometry course is billed as an encounter with the mathematical work of proving theorems. Outside geometry the subject matter itself is rarely organized around propositions that could be proved or falsified, and as Wu (1996) reports, "outside geometry there are essentially no proofs" (p. 228).

The high school geometry class has thus seemed to be an appropriate case where to do a systemic study of mathematics instruction centered on understanding what regulates the place of proof and proving. American students take the geometry class in either 9th or 10th grade usually depending on when they took their first course in algebra. The surprising thing about the geometry class, however, is not just that proofs occur almost only in geometry, but also that the proofs that do take place in geometry, particularly the proofs that students get to do, appear to a mathematically educated observer as somewhat odd. In Herbst (2002a) I analyzed a situation that is quite common in geometry classes, that of a teacher engaging students in the proving of a mathematical proposition. The analysis uncovered some characteristics of the work of a teacher engaging students in proving that might strike an observer as odd. One of them is the pervasive presence of the "two-column" format of statements and reasons (for an example, see Herbst, 1999, 2002), not just as a way of displaying proofs but also as a way of organizing the production of proof, eliciting in sequence the elements that go into a proof. Another one is the apparent need to hide the conceptual richness of a geometric question whose answer might involve students in giving a proof; and instead make relatively obvious what are the specific ideas that must be used as statements in the proof through the task statement or the accompanying figure. An observer of a geometry classroom might easily come away with the impression that students prove utterly uninteresting, and oftentimes obvious, geometric propositions, and they do so with an impressive attention to unnecessary formal detail.

And yet there is a way in which those observations only appear to surprise the mathematically educated observer who is not sufficiently familiar with the geometry course. In Herbst (2002) I show why and how geometry instruction took on the responsibility of holding students accountable for proving; in particular I show how the allocation of a stable share of labor for students involved a transposition of the notion of proof (Chevallard, 1991). This process of transposition began shortly after geometry became a high school course when the notion developed that students should not just be held accountable for re-producing the proofs of the theorems in the Euclidean canon (because such re-production was often discovered to be a memoristic reproduction). The notion developed that students should be in charge of solving problems that provided them with opportunities to prove new propositions that extended the knowledge from the text. For some time the development of students' capacity to prove propositions was an implicit aim of the interplay between theorems proved in the text and other propositions chosen for students to prove. Yet during the time close to and especially after the report of the Committee of Ten, the teaching of "how to do proofs" became more and more explicit - both in the way texts presented the proofs of theorems and in the way exercises for students were formulated and placed in the book. Near the beginning of the 20th century, texts also provided descriptions or definitions of what a proof was that provided a rather reductive view of proof as a formal, logical process and created the impression that what had to be learned did not have much to do with creating mathematical substance. Mathematics educators of the time, such as David Eugene Smith (1911), actually insisted that the point of having students do proofs was not to engage them in finding new things but in using what they already knew. The point was not for them to prove strong theorems but rather simple ones, precisely the ones that we might judge not mathematically interesting to be stated or proved at all, in which they would encounter success while practicing deductive reasoning. By the mid 1910's the two-column format of statements and reasons had already become a standard for enforcing those formal characteristics. Thus the historical development of a stable share of labor for students over proving propositions has been concomitant with the making of "proofs" as an object of study, an object defined almost exclusively in terms of its logical form though its relevance has been argued in terms of its substantive function.

Engaging students in proving: Who has to do what and when

With my research group (*Geometry, Reasoning, and Instructional Practices*, at the University of Michigan) we have been investigating geometry classrooms. Our archive includes longitudinal (year long) records from intact[3] lessons in several different classes covering the range of versions of the geometry course offered in one comprehensive high school. We have also collected artifacts, interviewed teachers, interviewed students, and implemented a number of experimental lessons.[4] The work done has converged among other things to a model of what we call the "normal"[5]

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[3] We call "intact" (alternatively, "naturalistic" or "ordinary") those lessons for which we have played no active role in planning the instruction.

[4] In general these experimental lessons (a.k.a., instructional experiments) are ones that we propose to and then negotiate a plan with teachers. I call these lessons instructional experiments because they often induce perturbations on some aspects of the customary instruction that takes place in these classes.

[5] By "normal" I mean neither "desirable" nor necessarily "expected" or "most frequent." By normal I mean "going without comment" or unmarked (in the same sense that IRE is the unmarked or normal pattern of interaction in situations of elementary classroom recitation; see Mehan, 1979): Situations that are described by the model are enacted by teachers and students without repair (or without negotiation of the didactical contract) and situations that do not follow the model require participants to engage repair strategies (or negotiate the didactical contract; see Mehan and Wood, 1975).

situation of students' engagement in proving. This model attempts to describe episodes in which students are held accountable for the production of a proof. By *model* I do not mean exemplary and do not necessarily mean a smaller-scale copy of something real. Rather I mean by model a simplified representation of a class of complex phenomena that can be used to describe how those phenomena occur and possibly make predictions about their functioning.[6] It goes without saying that these models are provisional intellectual constructions of an observer, not immanent characteristics of the actions observed. Furthermore, since these models describe what regulates human action, verbal reporting on them at times makes metaphorical use of mental state language (speaking for example of *tensions* or *dispositions*): What the models do, however, is to say that everything happens "as if" the actors meant to do certain things.[7]

The model of engaging students in proving that we have been working on includes two basic structures - an accountability structure that indicates who is responsible for doing what, and a chronological structure that indicates what must happen when. Like any attempt to model phenomena that are framed as "guided doings" rather than as "natural" occurrences - that is phenomena that actors perceive as constructed in part in response to their will (Goffman, 1974, p. 22) - any model needs to account not just for the regularities in the phenomena but also for the devices used in the phenomena to hide the appearance of such regularities. Bourdieu (1990, p. 98-111) illustrates this in the case of exchange of gifts, by suggesting that a model of that practice needs not only to account for the taking of turns and the equalization of values in the exchanges but also for how actors contrive[8] practice so that exchange appears as if no exchange was happening (as if every occasion of gift giving were a unique demonstration of appreciation rather than a pay back of a gift of equal value given before).

The diagram in Figure 1 shows a provisional model of the accountability structure of the situation of "doing proofs," which we have developed as a way to organize observational as well as student interview data (Herbst, 2002a; Herbst and Brach, 2004). The diagram provides a representation of the various aspects that need to be worked out so that students can engage in proving, and who needs to take care of each so that the occasion appears as normal. The model organizes accountability for "doing proofs" around six domains represented as white boxes. So there is accountability for the conclusion to be proved, for the initial conditions (or givens) that can be used in the proof, for stating that a proof needs to be done, for identifying or activating the concepts that need to be involved in the proof, for various kinds of interaction with the diagram (including creating it and marking it), and for various kinds of verbal or written actions in producing the proof. The boxes are placed in different accountability regions according to whether the teacher, the student, or none of them is accountable for what the box includes. The arrows establish connections between those accountability domains. So, for example, in the upper left corner the *activation of concepts* box is fully located in an area of high accountability for the teacher and linked with arrows to the *interaction with diagrams* and *problem statement*

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[6] See Schoenfeld (1999) for a similar conception of modeling.
[7] As von Glasersfeld (1995) aptly argues, this is the case in any modeling enterprise including of course the modeling of individual cognition, not a "limitation" associated to the modeling of social phenomena and by no means a suggestion that models of social interaction need to be reduced to more basic models of individual cognition in order to be viable.
[8] I thank Christina Hatcher for prompting me to think of this issue when she described the distance between elements of the model and real instances of engaging students in proving as involving original "maneuvers" by the actors.

boxes, to represent that the activation of concepts is done through either the given diagram or the statement of the problem. The *interaction with diagrams* box includes actions that differ in regard to who has major responsibility for them: to give diagrams with auxiliary lines drawn in falls under the teacher's accountability, whereas to read diagrams and mark in them the steps of the proof fall under the students' accountability. The proposition proved (by students) is placed outside the accountability region of either the teacher or the student. This indicates that neither the teacher is accountable to teach this proposition nor the student is accountable to learn it and use it later. Whereas the activity of proving ends when students prove such proposition, what they prove is normally not a useful or memorable result.

The diagram shows that students' share of labor when they engage in proving consists of writing statements and reasons as a demonstration of their capacity to reason logically and communicate clearly. Yet they do not take responsibility for crafting the statement they have to prove, for finding the conditions on which a statement can be true, for drawing extra elements into a diagram in order to produce a proof, or for activating the concepts that are instrumental for doing the proof. Furthermore, they tend to see the actual conclusion that they prove as of little importance usually regarded as obvious from a fairly drawn diagram and of no use for future work.

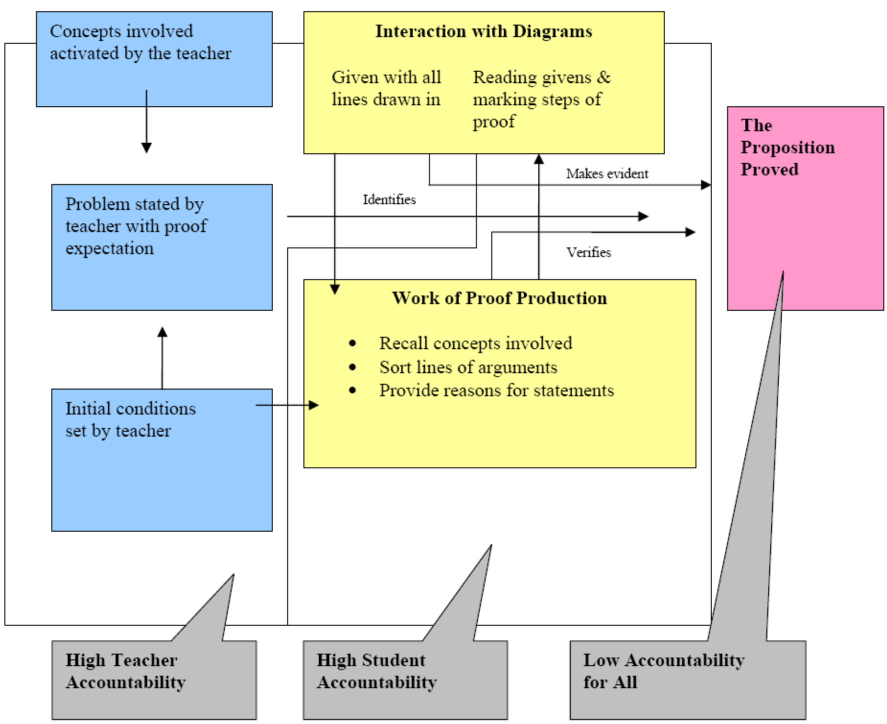

Figure 1. Accountability structure of the proof situation

As far as the chronological structure of the model goes, my observational work (Herbst, 2002a) puts forward the hypothesis that the two-column proof format - which is ubiquitous in high school geometry in the US (Sekiguchi, 1991) - is not just a format for recording reasoning but rather one for producing it. Insofar as it is a production mechanism it contains the resources for the enactment of a basic chronological structure that consists of four basic elements. The first component of this chronological structure separates a time[9] for setting up a proving task - including the creation of a finished diagram - from a time for proof production when no new additions to a diagram are done. In this second time, the chronological structure identifies different "moments" in the production of a proof - a time for producing a statement (usually a simple assertion about a diagram) or a time for producing a reason (usually a general proposition that has already been established in the course). Each of these events - the production of a statement that can be concluded as the next line of a proof, or the production of a reason for a given statement - is a moment of production, normally enacted in public by one speaker, usually a student. The grammatical and functional characteristics of each of those speech turns also establish some expectations of duration of a moment in that chronological structure: Normally speakers utter a short assertion or an assertion's shorthand (e.g., SAS, CPCTC[10]) and then end their speech turn.

A second element of this chronological structure facilitates the passing of time in a fashion similar to the triadic dialogue of initiation-reply-evaluation (Mehan, 1979) but specialized for the use of the two-column format. The time of production of a proof is usually paced by actual utterances, usually by the teacher but could also be internalized by the student, to the effect of saying "what can we say next?" or "why is that?" which control what kind of production moment comes next. This articulation between moments (added to elements of the accountability structure, such as the activation of concepts) also controls the extension of time that can be spent in between moments: It keeps pauses between statement and reason and between a reason and the next statement relatively short (in real time).

A third element consists of devices to begin and to end the production - usually the proofs that students do begin with a restatement of the "given" or initial conditions, and end with the reason for the statement that "could be stated next" and was equal to the "to prove" in the statement of the task. Hence if a teacher holds herself accountable for stating the task as one of doing a proof, and identifies what has to be proved in reference to a diagram, she is also providing transparent means for students to recognize "when" they are done. This can be much more opaque if the task is not stated in reference to a diagram or if students are charged with stating what is to be proved. A fourth element consists of devices to access previous moments in the production of the proof. Proofs often contain numbers for lines, to facilitate indicating that a new statement follows (or can be concluded) from a statement that is not (just) the last one.

As noted above, whereas an observer's description of the accountability and chronological structures may characterize normal episodes of "doing proofs" as accomplished through the enactment of a mechanism, the regularities of that mechanism are often masked in practice by maneuvers that aren't always breaches of normality. Participants maneuver discourse and timing in ways that permit them to

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[9] This use of *time* is in the *kairos* sense—a time for this and a time for that (Erickson, 2004).
[10] CPCTC, ubiquitous in American geometry classrooms, is a shorthand for the definition of congruent triangles and means that corresponding parts of congruent triangles are congruent. Similarly SAS and SSS stand for side-angle-side and side-side-side, respectively.

both know what they are supposed to do but also create the impression that what they are doing is, to some extent, novel. Inserting commentary between the lines of a proof or asking for alternative ways of doing the same thing (e.g., could you have done this using SSS instead of SAS (see footnote 10)?) are some of the ways in which a teacher could maneuver the production of the proof so as to make it appear novel. Breaches of the norm have a different flavor in that they are accompanied by "repair strategies." For example a teacher who, going over a proof, starts with a first line that is not the "given" might also say "I don't always need to start with the 'given'." Needless to say all these issues are still the object of research - particularly as regards to the perspective of the teacher.

If such model of a normal situation of engaging students in proving is plausible, one gets the impression that the place of proof in high school geometry classes differs considerably from the place of proof in mathematics vis-à-vis the development of knowledge. The results of interviews of students-as-informants are in agreement with those of the historical (Herbst, 2002) and the observational (Herbst, 2002a) studies in supporting that description of the division of labor in regard to students' engagement in proving. They also confirm previous observations documented in the literature, for example in works by Schoenfeld (1987, 1991) and Lampert (1993). All of these studies point to a particular aspect in which the mathematics of classrooms seems very different from that of mathematicians: Proof plays a somewhat marginal role in finding or establishing truth - it is rather isolated as an exercise in logic and communication with little bearing on making sense. Geometry students are socialized into a custom of proving that not only supports their engagement in proving, but also precludes their investment of some cognitive actions most germane to mathematical reasoning; in particular *proving* is dissociated from *coming to know*.

However we should be careful in the way we spin that observation. A voluntaristic reading of that observation might compel us to say that since classrooms are so wrong we need to fix them. Yet my contention is that if that model describes anything real, what it describes is only a symptom of something bigger regarding the function of instruction that we still need to understand - that is, we need to explain on what grounds it is reasonable (even if undesirable) that it happens. An understanding of the systemic causes of these phenomena can help figure out what changes are viable and what side effects they can create. In the spirit of finding an explanation, I suggest that the particular ways in which proof exist in geometry classrooms are actually a symptom of the internal work of an instructional system that has basically succeeded in allocating a stable place for the activity of proving though at the expense of making proof an object of teaching and learning.

Searching for plausible instructional explanations

The phenomenon that we need to understand better is the apparent alienation of practices of proving from practices of knowing and making sense -which include both the discovery of new knowledge and the solving of problems. We want to understand why an instructional system would produce such performances. Later on we may be able to envision how that system could be changed in order to produce different outcomes. The rest of the discussion introduces some elements of theory that I have found useful for thinking about this phenomenon.

Students and teacher interactions related to proving and proof, like many of their joint experiences with mathematics in classrooms, are complex because they take place in a multilayered context. One layer of context is created by particular *tasks* that involve students in doing some *mathematical work*, reasoning about mathematical ideas at play in specific questions. Another layer of context is provided by the institutionally situated, triadic relationship between teacher, students, and knowledge-to-be-taught and the implicit didactical contract that binds them. Let me elaborate first on what I mean by *contract* and what I mean by *task*. I will then propose that the existence of these two layers of context for interaction presents a problem that participants need to deal with in order to be able to interact.

Classroom mathematical work exists within an institution - the school with its various stakeholders. Teacher and students interaction about mathematics exists because an institutional relationship vis-à-vis the study of mathematics makes it possible. They don't just gather spontaneously and they don't stay there because they like to be there. Their interaction is possible because of the institution and does not come with a blank check; rather it operates on the assumption that a *contract* exists between teacher and students vis-a-vis mathematics. This contract is what Brousseau has called the *didactical contract* and that English-speakers might like to call *instructional* contract. The contract is a mostly implicit, statement of global, mutual responsibilities between teacher and student as it regards to the subject of studies. Like most of the constructs and concepts we use in social science, the contract is not an actual legal document but a hypothesis of the observer that helps understand human action. The notion that a contract exists between teacher, student, and subject of studies helps one understand why teacher and students act as if they had permission, but were also under the obligation, to do some work together that they need to ensure is mathematical, to make claims concerning mathematical knowledge, to divide labor and to trade that joint work for claims on the mathematics that they presumably "know," "have taught," or "have learned." Among the few elements of the didactical contract that are explicit one finds, at least in the US, some level of specification of the course of studies - there are specific things that students come to a course to learn, things that teachers of a course must teach, things that students leaving the course must know, and that articulate the way in which they talk about the mathematics of classroom interaction. I call those objects, contractual *objects of trade* and suggest that in US geometry classes *proof* is one of those contractual objects of trade, as much as equations is so in US algebra classes. At least since the beginning of the 20th century (Herbst, 2002) proof is something that the geometry class has been responsible for teaching and learning—the key obligation on whose account the geometry course was sustained in the curricular debates of the end of the 19th century.

Teaching and learning themselves are activities which are also (at least implicitly) regulated by a contract. Whether we like it or not, classroom interaction is possible most of the time because knowledge is seen as a commodity that people can posess, that teachers can transmit, and that students can acquire within given timeframes. The existence of a didactical (not just a pedagogical) contract not only ensures interactions over knowledge, but also biases the meaning of *teaching* and *learning*. Even though a teacher may be able to pull off practices that do not quite respond to that bias, the classroom does not only have an inside where people negotiate what they do; it also has an outside whose expectations the classroom must respond to. That compound is the contract. Changes of the didactical contract in that compound sense are not just a matter of happy agreement, it is not "what you would like to do today" and not just "how you set up your classroom at the beginning of the year:" If changes happen, they are tied to a historical rather than a day-to-day timeline. In that contract proof has been shaped as a skill - how to do proofs - but more importantly as an object that can be traded: Teachers teach *it*, students learn *it*, and at some point *its* teaching and learning ends (i.e., students know *it*). A natural question to ask within that economic metaphor is therefore, what is the currency?

One of the things that makes classroom interaction different than many economic exchanges is that its currency is not a thing like money but *work* - action in context over time. Teachers and students do mathematical work together and cash it into contractual objects of trade. I use the word *task* to refer to a semantically closed chunk of classroom interaction, a unit of meaning deployed over classroom interaction, a piece of work. For example the work done to produce one answer for a particular question is a task. The notion comes from Doyle's (1988) ecological approach for describing the curriculum as experienced by students - a task is an arrangement of agents, resources, operations, and products. Scholars have tended to use task as the statement of a work to do - yet the work of Stein et al. (1996) has shown the importance of looking at a task as enactment. In an effort to see task as enactment and to achieve semantic closure - that is to make it stand at least for the observer as the embodiment of a mathematical idea - I have found it useful to describe tasks as not just the question being answered, but also the operations used to answer it, and the resources available to answer it. The task (in the sense of a unit of enacted work) is different whether or not a diagram is used, whether or not certain previous knowledge is active, etc. no matter whether the problem is basically the same. Furthermore, work is being done all the time in classroom interaction - whether students are supposed to produce something or just to listen to something, joint work happens continually. Task is a unit for parsing that joint work that, at least from an observer's perspective, pays attention to mathematically distinct and autonomous chunks. The work done to produce an argument can be an example of a task.

So students and teachers engage in mathematical work. Doing that work is the way to *pay* for the objects of trade - to make a claim to knowledge of them. In particular some work done in the class can be cashed for the teaching and learning of proof. For example students' knowledge of proof is claimed, among other things through their successful engagement in proving propositions. Students' knowledge of proof, like knowledge of anything else, cannot be cashed by placing a check mark on a box, it must be deployed in work. In that sense trade is not like purchasing goods as much as it is purchasing some kind of service - such as getting "entertainment" from going to see a film or getting "excitement" from going to an amusement park with roller coasters. And it is demonstrated among other things through the making of particular arguments for specific propositions. The argument that proves a specific proposition counts toward students' knowledge of proof. Or does it?

Not every argument made for a proposition counts (in the view of the actors) as students' knowledge of proof. The extent to which one such task amounts to work that can cash for something (e.g. that can cash for proof) depends not just on the extent to which the work done answers the specific question of why a claim is true or what claim is reasonable to make. It depends also on whether participants in the class are in a situation in which they can identify that piece of mathematics as tradeable work and effect such cashing. Evidently not all ideas manufactured in classroom work are ever cashed - questions need to be asked as to which ones are and what they are cashed into. This calls for understanding the customary ways *for actors* to parse and recognize work and for cashing that work for knowledge in a geometry class. To be able to talk about this, I bring about a third construct: the notion of *situation*.

The point I want to make is that effecting a *trade* between *work* and *knowledge* requires a space, a marketplace of sorts, in which work done can be sized up and ascribed a public value, where work done can be seen as a unit and can be labeled on behalf of some contractual object. To designate those spaces that contain tasks and make it possible to cash one or more of them for a contractual object I use the word *situation* as a shorthand for *instructional situation*. The choice of the word follows the work of sociologist Erving Goffmann - who makes the point that analysis of talk is meaningless unless complemented by analysis of the situation in which that talk exists (Goffman, 1964/1997). The notion that a situation is a space for sizing up and cashing work into objects of trade is also useful for the analysis of intact as well as experimental classroom interaction. Situations are thus spaces for identifying and cashing work for knowledge. From the inside they are containers of work that an observer might see as including one or more tasks. From the outside they appear as units of work that can be filed or labeled on behalf of some contractual object. Situations are of the same grain size as what some call activity structures or patterns of interaction (see Lemke, 1990), but they are not generic with regard to knowledge. Situations are units to parse interaction in mathematics classroom, but of a different nature than that of the task or that of the contract. Situations have accountability and chronological structures - scripts of sorts that regulate who is supposed to do what and when; they may instantiate patterns of interaction or activity structures as they play out. What is important about them is not how they organize talk or interaction but how they use that organization to effect the trade between work and knowledge.

The US high school geometry class provides a paradigmatic example of situation *a propos* of proof and proving in the form of what I have called the “doing proofs” situation, whose provisional model I offered earlier. The doing proofs situation consists of a customary space (marketplace) for enabling certain kinds of work to be identified as proving and cashing them for a claim about knowledge of how to do proofs. I started this discussion by noting that the place of proof in high school geometry classrooms is somewhat odd when one looks at it from the vantage point of the place that proof has in mathematics. The framework just presented provides the bare bones for an explanation of why that happens that restores rationality to the practice and its agents.

Instructional situations make room for cognitive tasks to exist. They create public space for some work to take place and be interpreted on behalf of certain objects of trade. But in doing so, they “normalize” those tasks. Certain cognitive tasks may be conceivably related to some mathematical ideas - because a person’s involvement in such task might embody those mathematical ideas from the perspective of an observer. Yet the viability of those tasks in classroom activity depends on instructional situations that can provide them with context. In doing that, situations constrain their potential to embody those ideas. In many classrooms where we have done instructional experiments the following phenomenon takes place: A mathematically interesting task is enacted, particularly one in which students craft a mathematical argument to answer a substantially valuable question, yet the label that such work gets, if the work is visible at all, is seldom one that accounts that argument as proving - other situations, such as exploration or discussion, are the ones that make room for that work and set their value.

Our data from interviews with students and teachers as well as from observations of work in intact classrooms, on the other hand, show that though a range of tasks might conceivably embody the meaning of mathematical proof from an observer’s perspective, the situation of “doing proofs” normalizes that range into a narrower set of tasks. This ‘normalization’ makes tasks viable (within instruction) by ‘rewriting’ them according to some standardized accountability structures that are in place to facilitate their accomplishment and their cashing into objects of trade (elements of the knowledge-to-be-taught—an official notion of proof-as-deductive verification of facts). If most proving tasks are logical puzzles this is not only a shaping factor but also a consequence of the curricular notion of proof as a notion that privileges form, logic, and procedure over the conceptual substance and strength of connections.

Practical rationality

The notion of situation is a tool that helps understand why things happen in mathematics class. It helps understand for example that the notion of "doing a proof" is not just proving a proposition but also showing that one knows "how to do a proof." Furthermore it helps understand how, for participants to be able to do both things at the same time, they may need to do something different than just engage in proving. And indeed they may do things that endow practices of proving with a meaning much more restricted than the meaning proof has in mathematics. Not all tasks are created equal, at least in relation to their chances to engage students in proving - but the more a task has such a chance, the less well the doing of the task serves the purpose of finding out. The foregoing argument explains how that phenomenon is not so much an indication of lack of rationality. Rather it is an indication of how instructional practices have historically developed a rationality that allows them to adapt to the need to teach proof and have students produce proofs. This takes me to a last point.

My previous analysis has privileged the perspective of an observer who, among other things, is able to see mathematical tasks partly by virtue of being detached from action. That perspective appears to provide, at least at first sight, a somewhat static view of classroom interaction. The approach makes the point that things that happen in a class respond less to a lack of knowledge than to the having of specific (though implicit) knowledge. But it fails to describe, yet, how that implicit knowledge operates strategically, as intelligence in action to construct viable practice. How does that implicit knowledge regulate what actors of a practice perceive as viable versus what they perceive as not viable in engaging students in proving? Tapping into that intelligence is valuable. After all, in spite of the grim mathematical picture that one sees in many geometry classrooms, amazing things also sometimes happen, and one would like to be able to say something as to what are the conditions in which it is possible for situations of engaging students in proving to allow for more authentic mathematical activity.

A big part in understanding that depends on understanding the so-called wisdom of practice of the practitioner, or what, following an expression of Pierre Bourdieu, we have called the "practical rationality" of mathematics teaching (Herbst and Chazan, 2003). A geometry teacher is very much like a skillful player of a game; a game which is only partially understood by modeling its objective rules. The geometry teacher, like a soccer player, has a *feel* for the game (Bourdieu, 1998). That feel for the game we need to capture in order to understand better what could be the place of proof in geometry classrooms. Like in the case of other competent practitioners, soccer players, chefs de cuisine, artists, much of that feel for the game is implicit in action, not deployed in any naturally occurring discourse and seldom retrievable from decontextualized interviews.

Furthermore, much of that rationality that regulates viable instruction is held not as much as beliefs or commitments of individual practitioners as they are *embedded in the work they have to do*. This rationality comes with the job, so to speak, it is adapted and adaptable to the specific work that teachers have to do, and amenable to transformation as their work evolves. As a provisional definition we can therefore say that the practical rationality of mathematics teachers is the system of implicit dispositions that regulate how teachers create responses to the obligations that tie them in an institutional context to their students and the course of studies they teach. Teaching students how to do proofs is one of those obligations, the situation of "doing proofs” is one key context in which teachers fulfill that obligation and where they employ their practical rationality.

In our current work on the project “Thought Experiments in Mathematics Teaching” we explore a methodology to elicit the practical rationality of mathematics teachers - specifically of teachers of high school geometry and high school algebra. The project will create representations of teaching episodes and use those representations to display alternative ways in which the situation of “doing proofs” could unfold in a geometry classroom - alternatives that range from the normal (i.e., episodes that embody the model of normal situations described earlier) to the marginal (i.e., episodes that embody various possible deviations of the model of normal situations described earlier). We attempt to use those representations as context for conversations among groups of geometry teachers which we expect will bring forth teachers’ practical rationality as they argue why certain courses of action are more or less viable than others.

Conclusion

Since Piaget, and through the immensely valuable work of mathematics education scholars who focused on the study of learning, our community has naturalized a perspective on research that in a way breaks with the illusion of immediate knowledge. When students act in idiosyncratic, erroneous ways, we see in that not anymore a sin, or a slip of the pen, or plain forgetfulness. We don’t see students’ performance as indication of a lack of knowledge as much as an indication of their having of a different knowledge. We see a symptom of a rationality that we have to uncover and understand. We have learned to restore rationality to the child. But individuals are not the only cases of emergent knowledge, organizations are so too. And classrooms are such organizations, complex ones. In classrooms, teacher and students are not just freelance agents - they play roles, they fulfill expectations, they interact and they transact cultural goods and shared work. In doing so, like the musicians of an orchestra playing each their own part, they create public knowledge - a public image of mathematics. That public knowledge is the most immediate opportunity to learn for individual students, hence something that is not just a concern for idle speculators. Yet, like the performances of a single child, the performances of a classroom can be mathematically odd. I vow that what we have learned about individuals will make it easier to recognize that those oddities are not sins, or mistakes, or slips of the pen. They are not just indications that the actors lack the knowledge or the skills needed to do what we think they should. Those oddities in instructional practices are symptoms of a practical rationality, the rationality of a practice, that we have to uncover and understand. Mathematics instruction is a social practice that, like many others, requires implicit regulations in order to function in a stable way - those regulations, like the regulations of any social practice, may allow some things and exclude others, and it matters very little whether outsiders like such inclusions and exclusions. Responsible instructional improvement needs two cups of internal critique for each cup of activism and voluntarism.

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