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A crucial issue in mathematics education: The ability to change representation register

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Introduction

Research in mathematics education is mainly concerned with teaching, and the learning process is analysed in relationship to teaching conditions. For example investigations into the problems that favour the students' activity and how far the interactions between students in solving problems make them become aware of what they must learn. However the analysis of the learning process involves an assumption about a question which is rarely asked: are the ways of thinking the same in mathematics as in other areas of knowledge?

This question about the cognitive process underlying mathematical activity can seem strange because ***what is common*** *among* ***mathematical knowledge*** *and the others areas of knowledge* is always emphasised, and applying mathematics to situations of the real world leads to such an orientation. But very often most learners, at every curriculum level, are more sensitive to something like a hidden gap between the mathematical ways of thinking and the ways of thinking outside mathematics, even if mathematical knowledge can be used in real world. Are they wrong? Anyway teachers often observe that the acquisition of some mathematical knowledge does not make most students enter the mathematical way of thinking. And the ability to change representational register belongs to this way of thinking.

In order to understand what occurs as a recurrent source of trouble in learning mathematics we need extensive and detailed investigations about the cognitive processes involved in the mathematical way of thinking. To introduce the framework of such an investigation I will start from two common observations about the wide range of "representational contexts" in which mathematical knowledge objects can be given.

- Mathematical activity is necessarily done in a "representational context". Thus for numbers there are the quasi-material representation with matchsticks (IIIII IIIII), the polygonal representation and also the decimal notation system with this very strange sign **"**0**".**
- But students also must be able to recognise the same mathematical object of knowledge in other "representational contexts" and to use them.

What is the role played by these inescapable and heterogeneous "representational contexts" in mathematics comprehension and learning? They are often interpreted as products occurring either at the surface level of mathematical activity or at the terminal stage of the thinking process. In others words they would be only external and further away from mathematical comprehension, which should be mental or purely conceptual. Is such an interpretation correct?

In fact we must be more precise in these common observations. The representational contexts which are necessarily used in mathematical activity are semiotic**.** Taking into account the semiotic nature of representations used in mathematical activity means to take into account both the way they are used and the cognitive requirement involved.

(1) What matters is their property of *transformation* because *mathematical processing* always involves some transformation of semiotic representations. The primary role of signs in mathematics is not first to stand for objects but to be substituted for another signs like, for example, in calculation. Moreover this property of transformation *depends on the semiotic system* of representation within the representations that are produced. In that sense there is not a "semiotic mediation" but quite different "semiotic mediations."

(2) Mathematical activity requires from the individuals the possibility to use various semiotic representation systems (representation registers), even if only one is explicitly chosen according to the purpose of the activity. In other words mathematical activity requires the development of an *internal co-ordination* between the various representation systems which can be chosen and used. Without this internal co-ordination, which has to be constructed, two different representations mean, for students, two different objects without any relation between them, even though they are two different "representational contexts" of the same object.

These are like the two sides of mathematical activity. They cannot be considered apart from each other, especially when one attempts to understand the problems of mathematics learning. They provide the key idea for analysing the cognitive processes which are involved in mathematical thinking. From the first item it is obvious that two kinds of transformations of semiotic representations must be distinguished: *conversion* and *treatment* (Duval 1995a). As to the second item we can examine the cognitive complexity of the specific kind of transformation which requires a change of the semiotic system used while a mathematical activity is started or in its progress. Hence the issue: why so much recurrent trouble about the representational conversion, and how to make students enter the functioning of representational conversion in mathematics ?

Two kinds of transformation of semiotic representation

When we analyse any mathematical activity we have to distinguish first between two kinds of transformation. To introduce them let us start with three examples.

TRANSFORMATION from one semiotic representation into another one

CONVERSION **TREATMENT**

Changing the semiotic system (register) ***used without changing the objects being denoted***

John is 3 years older than Peter. Together they are 23 years old. How old are they?

↓

x + (x + 3) = 23

Keeping the same semiotic system

x + (x + 3) = 23

↓

2x + 3 = 23

↓

2x = 23 - 3

....

Figure 1

In this very elementary example conversion corresponds to one single transformation of representation while treatment corresponds to a sequence of several transformations. But most often conversion and treatment are completely interwoven in the same mathematical process of problem-solving.

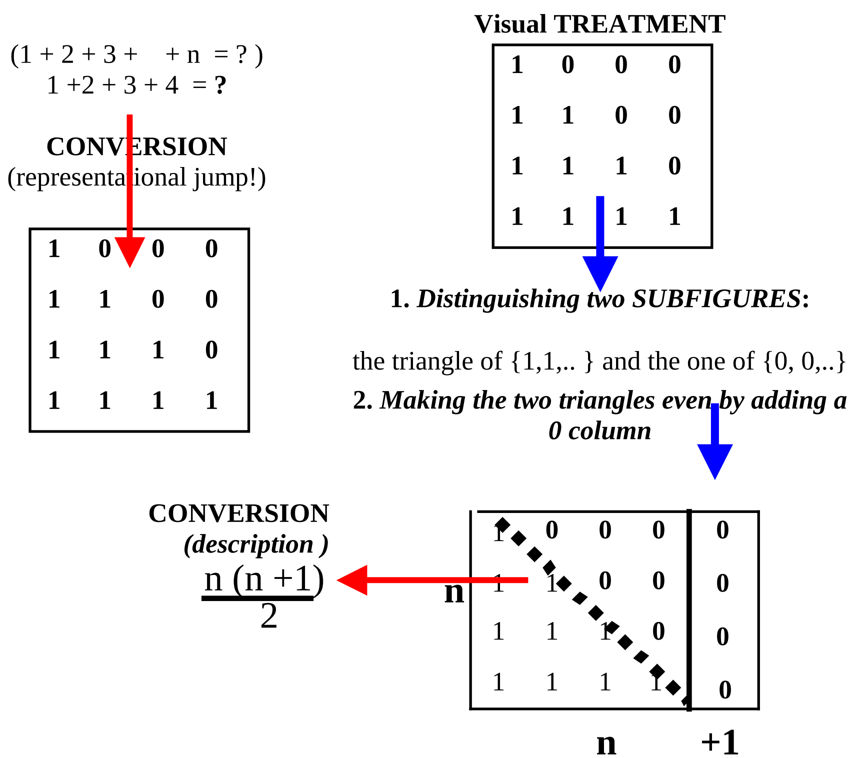


Figure 2

In the first example, conversion seems easy because it looks like a simple coding. But if we change the problem then the conversion process may become much more complex.

The transformation of a linguistic statement into an equation hides two specific requirements. Firstly one must use fewer symbols than objects to be referred to. For that one must construct a new expression by using an arithmetical operation. Then one must establish explicit relation in order to translate the meaning of the sentence into the equation. Thus we do not get the same semantic segment of problem data within the linguistic statement and within the algebraic statement. This is a first jump. But there is also a second jump. Within an algebraic treatment symbols of operations prevail against symbols standing for numbers. Expressions standing for numbers are therefore broken up.

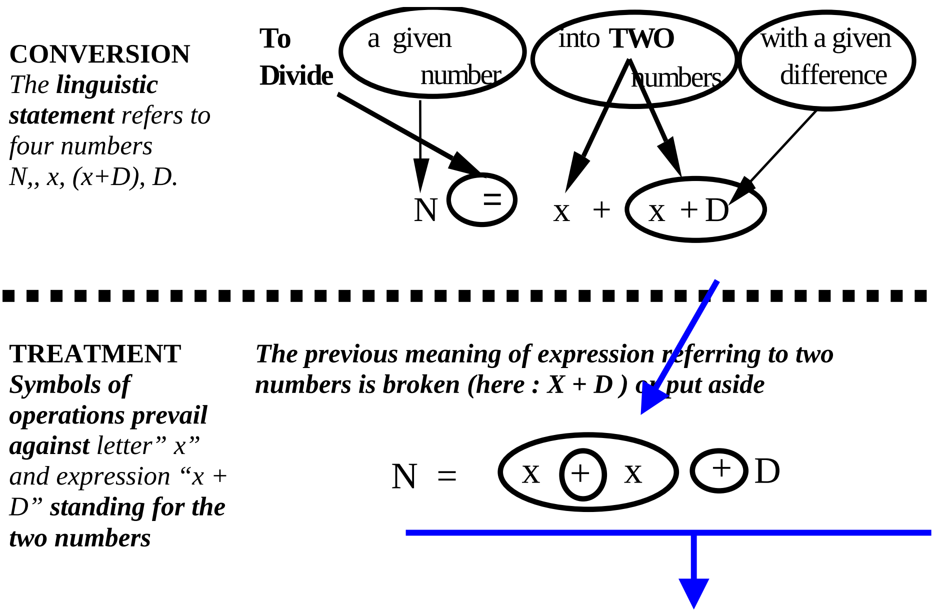


Figure 3

The complexity of this double jump is often misunderstood in teaching. Thus we can read in a national statement on mathematics claims like this one: "it is essential that students understand that letters stand for numbers, not for objects." Yes, but which numbers? In relation to the linguistic statement, letters within equations refer already to numbers (upper row arrows in Figure 3), but this first meaning of the letters must be broken up in order to start the treatment (lower arrows in Figure 3). Anyway it is only part of the complexity of conversion and not the most important one (Duval 2002).

In these examples of conversion and treatment two kinds of transformation of semiotic representations, which can be clearly identified, appear. They occur like separate stages in the processes of problem solving. But there are also situations in which conversion is implicitly and continuously required whenever two registers of representations must be mobilised together in parallel. The most typical examples are in geometry where these two kinds of treatment are very often required. One runs in a discursive way by valid deduction of properties from data and theorems which involves using language. The other runs in a visual way by the play of various reorganisations of shape (Duval 1995b). The processes of these two kinds of treatment operate separated from each other, because they do not mobilise the same cognitive systems. However, mathematical activity in geometry depends on their cognitive interplay. Treatment within one register can be aimed at or controlled by what is asked in the other register. This cognitive interplay requires the possibility to make local connections explicit. Encoding the visual representations with letters or marks helps just to explicate the anchor points for mathematical discourse within the various possible shape organisations. The mathematical common usage of the word "figure", which leads to confusion of visualisation with its encoding, also leads to misunderstanding the specificity of these two kinds of transformations running independently, as well as the complex role of conversion underlying any geometrical activity.

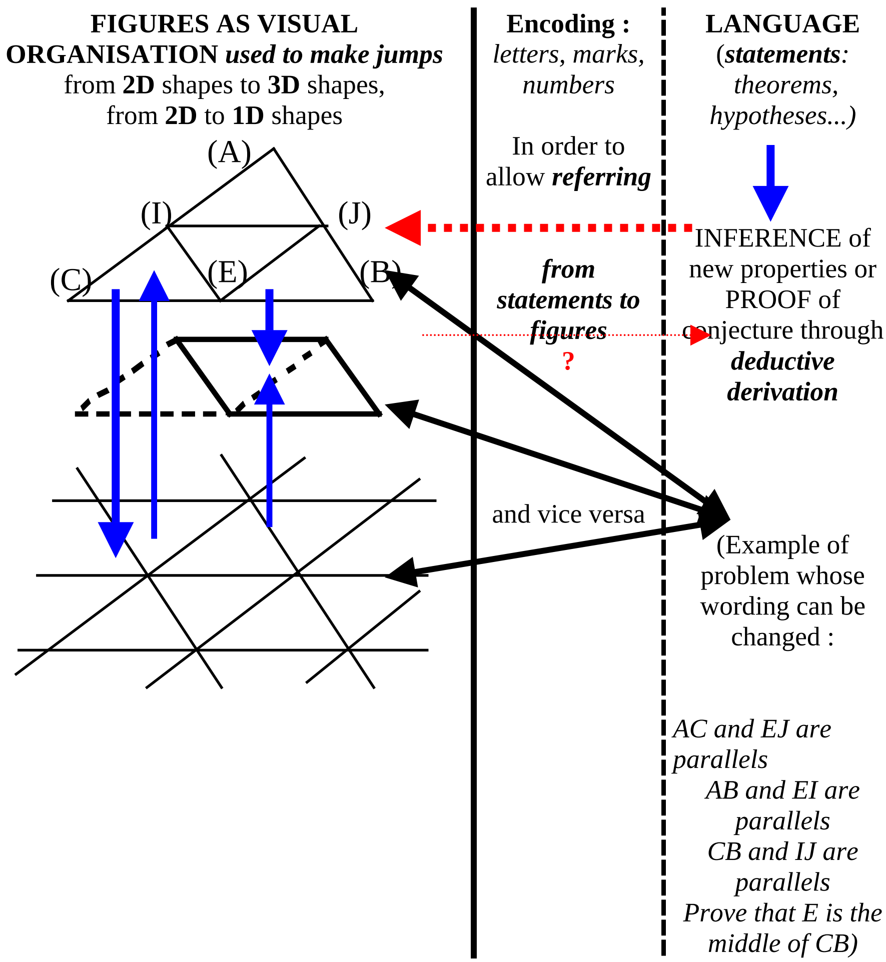


Figure 4. The complex process of conversion and two kinds of treatment in parallel underlying any geometrical activity

These two kinds of transformation, conversion and treatment, merge into a single geometrical process, even if one treatment prevails over the other (Duval 1998). However they remain two quite different cognitive ways of functioning, which depend on independent representation systems.

All these examples highlight that these two kinds of transformations of representations lie at the heart of mathematical activity. It is the reason why the first methodological requirement for analysing the problems of mathematical learners' comprehension at each stage of the curriculum, is to completely separate these two kinds of transformations. Such a claim may seem strange.

From a mathematical point of view, conversion and treatment make up a whole as regards problem solving. Furthermore, what matters is treatment, because it is the treatment to be carried out which makes the choice of register shift relevant, according to the "best" one (low cost, more suitable for generalisation, or more intuitive...) for solving the given problem. Lastly, would understanding and misunderstanding of concepts not be more basic than semiotic representations?

But from a cognitive point of view, things work out otherwise. Conversion and treatment are quite independent sources of problems in learning mathematics. And conversion appears to be a more complex cognitive process than does treatment. The trouble that most students encounter is so deep that conversion can be considered as the *threshold* of comprehension. Conversion of semiotic representations often appears as something of a magic trick that cannot be truly learned and which is not taught!

The cognitive complexity of conversion

Shifting the representation of mathematical relations or objects from one semiotic system to another is always a cognitive jump. Unlike treatment there are no rules or any basic associations, like those between words and images for everyday language, to help make this kind of transformation. Conversion is not a matter of coding. Lets us examine what could be put forward as a counter-example: the cartesian rule of coding for plotting any cartesian graphical representation of an equation or an inequality.

This rule gives only associations between points and pairs of numbers, which only allows for a selective numerical perception. Thus we can "read" a graphical representation. However, using this rule for plotting any graphical representation cannot lead to noticing the visual features which correspond to the characteristics of the algebraic equation converted, because these visual features are qualitative and global and not numerical and local. In order to get evidence of this cognitive gap one has only to give a choice task in which the usual direction of conversion, which focuses on numerical aspects by plotting and reading, is inverted (Duval 1996).

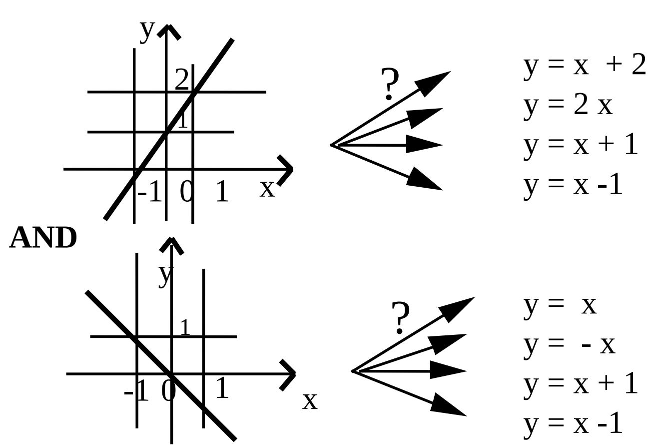


Figure 5. One single item of a qualitative recognition task for conversion between graphical representation and algebraic notation

When students are faced with this kind of a qualitative recognition task, it becomes obvious that for most of them the ability to discriminate the relation which is visualised cannot follow the practice of plotting and reading numerical values on graphs. And that can be checked even after functions have been taught. In these conditions how can all graphical representations get visualisation power, or give an intuitive support, to most students? These representations are absolutely no use without a fluent recognition of the visual features of curves which are mathematically relevant, except for reading a couple of numbers and identifying a point. Hence the basic question for any learner. *How to discriminate those visual features of the graph that are mathematically relevant for conversion?* In other words, how to see the semantic characteristics of an equation through the qualitative visual features of a drawn graph and vice versa?

To solve this problem we must take into account the way in which the semiotic system of cartesian representation can represent mathematical objects (relations and not only functions). The basic law of semiotic functioning is the following: Anything cannot function as a representation outside of the semiotic system *in which its meaning takes on value in opposition to other representations within that system.* By applying this, we get the following network for the visual features which are mathematicaly relevant whithin this kind of representation:

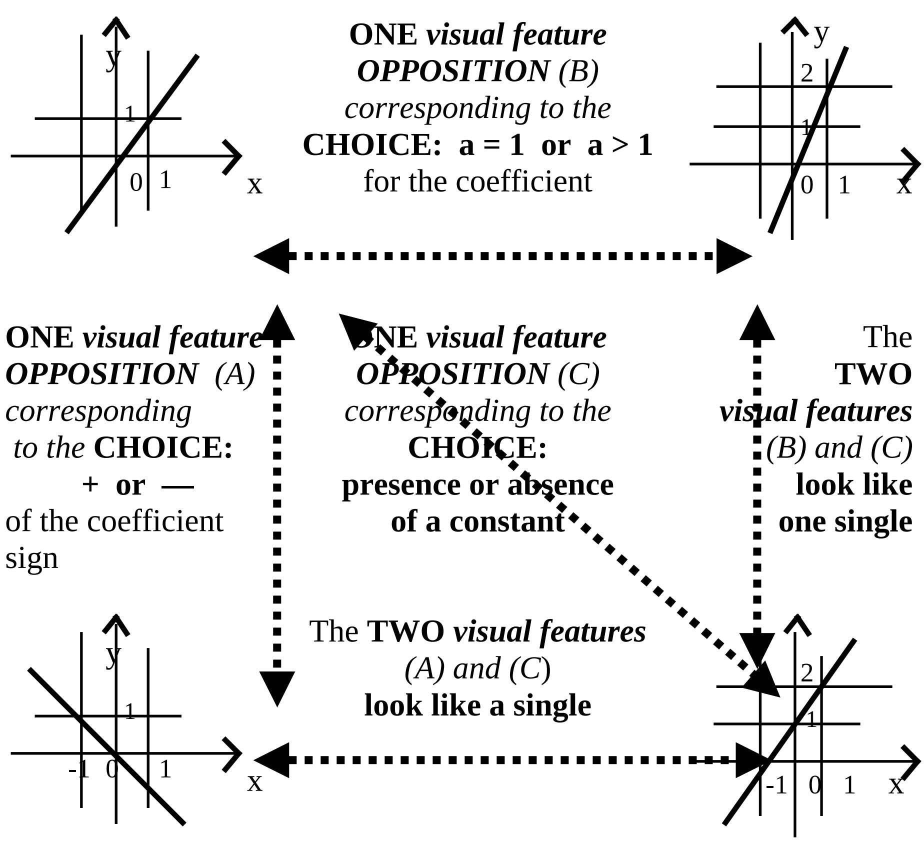


Figure 6. Network of the cognitive discriminations required for a conversion between graphs and equations

We shall make four remarks whose the consequences are important for mathematics learning.

(1) Each distinctive visual feature can be distinguished *only through the opposition of two graphs.*

(2) Each distinctive visual feature *matches with a semantic characteristic of the equation and* not *with the function represented.*

(3) It is on the basis of such a network of distinctive visual features that students become able to convert graphs and equations fluently and meaningfully.

(4) Such a network can be extended to all kinds of function representations and also to representations of relations and curves which are not functions. That means *such a network does not depend on some specific mathematical content or "concept"***.**

Now can *we assume that students become naturally aware of all these qualitative discriminations* only by plotting and reading graphs and by acquiring the "concept" of linear function? This is very easily assessed by means of recognition tasks.

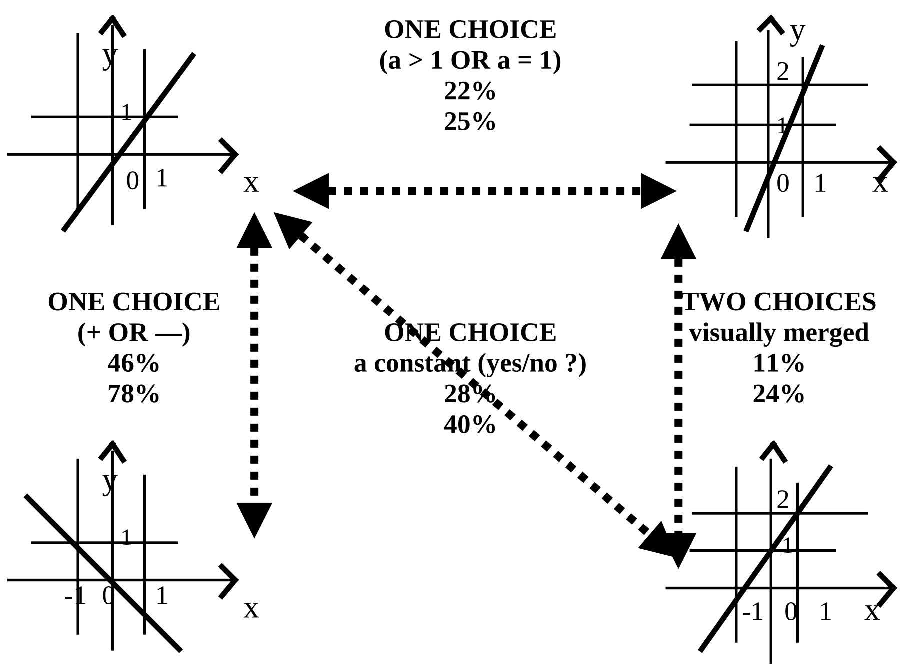


Figure 7. Recognition tasks involving the distinction of relevant visual features (the lower figures: after three-months teaching focusing on linear functions with 15-16 students)

Let us back to the first example (Figure 1 and 3). Conversion from a statement of more than one clause to one equation involves two levels of quite different discursive operations. One is cognitively more complex than the other, which results in a drop in success rate. To make this obvious we must break down the problem into tasks corresponding to both levels.

A man is 23 years older than is son. He is 31 years younger than his father. The sum of the three ages is 119 years. If x is the age of this man x + 23 refers to the son's age: YES NO x - 23 refers to the son's age: YES NO	Right choice of (x-23) *and* of (x+31): **82 %**
What equation allows us to calculate the age of this man ?	Organisation of these expressions into an equation which corresponds to the statement **46 %**

Figure 8. (involves 14-15 students)

In this example the conversion appears under its most elementary and congruent aspect. But in most cases the given linguistic statement is more complex and can also be requested to be converted into a system of equations.

We could vary the situations. Whatever the kinds of start register and of target register (language → symbolic notation, figure→ language) it is very easy to highlight that most students are in trouble whenever a conversion is explicitly or implicitly requested in mathematical activity. For that we have only to use devices of tasks which are less global than mathematical problems. We can also investigate the various factors which work in favour or against recognition of the same mathematical content within a representation and its converted representation into another semiotic system.

However, the important issue does not lie there. It lies in understanding what cognitive processes are involved in mathematical thinking and why the complex cognitive process of conversion is crucial for learner's comprehension.

Why conversion is crucial for learners' comprehension

Signs are too often associated with conventional notations like the use of letters in geometry and the use of symbols in algebra. From this very restrictive point of view we get a dualist approach to mathematical activity. There is, on the one hand, the mathematical content which would be conceptual and non-semiotic, and there are, on the other hand, semiotic representations which can be chosen according to the needs of communication or according to the cost of treatment. The former would be mental and the latter would be external or material. From this point of view, conversion would be the outcome of conceptual comprehension, and any trouble with conversion would be indicative of misconceptions. Does such a modelling of cognitive processes fit with what can be observed of mathematical activity?

Mathematical activity must meet two conflicting requirements:

— (I) semiotic representations must necessarily be used even if there is a choice of the kind of semiotic representation

— (II) the mathematical objects represented must never be confused with the content of the semiotic representations which are used.

The first requirement is especially important in mathematics for two reasons which highlight the particular epistemological situation of mathematics. Unlike the other scientific areas (astronomy, geology, chemistry, biology) knowledge objects (numbers, functions and their properties...) are not accessible from physical data, through sensory evidence or by using instruments. The only way of having access to them and of dealing with them is by using signs and semiotic representations. However the necessity of signs is not confined to that. Their main role in mathematics is not first to stand for mathematical objects but to constitute a platform for working on and with them, by substituting some signs for others. Thus the main role of numbers representation system is not to denote numbers but to make calculations. And the algorithms for numerical operations are different according to the system used, decimal notations or fractional notations or even representation systems without the sign "0". Semiotic systems are mainly used for operations, i.e. for treatment. We can summarize that by saying: without "semiotic mediation" no mathematical activity.

The second requirement might seem more general and not specific to mathematics. But it is with this requirement that most students get into trouble. The content of any semiotic representation does not depend on concepts or on represented objects only, but also on the semiotic representational systems used. This is why shifting from one system to another means changing the content of representation without changing the mathematical properties represented. Then how can the mathematical content be distinguished from what is specific to the semiotic system used and is not of mathematical relevance? Outside mathematics this problem does not arise, because the main access to knowledge about objects is non semiotic, that means it is given independently of any "semiotic mediation". But in mathematics this is not the case, because semiotic representations are always needed according to the first epistemological requirement.

In these two conflicting requirements lies the cognitive paradox of mathematics that most learners encounter in a deep manner. And that raises a deep problem of comprehension which is specific to the learning of mathematics. The most obvious touchstone for understanding is the possibility of *transferring* what has been learnt to new and different contexts, inside and outside mathematics. And this always involves representational conversion. Whatever the orientation of teaching being emphasised — applying mathematics to real-world problems or teaching mathematical concepts and procedures — most students are stopped at *this threshold of representational conversion*. For them there are as many different represented objects left as there are different representational contents used. A mathematical isomorphism between two representations never involves a cognitive isomorphism, and hence a fortiori cannot be recognized by learners.

Contrary to the dualistic approach in which the inescapable semiotic mediation would remain external and posterior to the conceptual comprehension, the semiotic representations must be taken into account in the analysis of mathematical thinking. Comprehension does not lie first in a jump from the content of a representation to the pure mathematical concept represented but in the relationship between various representational contents for the same concept. There is no one homogeneous "semiotic mediation" but several which have little or nothing in common. And as we have already seen (Figure 6) the representational contents depend not only on what is represented but also on the representation systems used. Therefore mathematical understanding requires of individuals an *internal co-ordination* between the various possible semiotic representation systems which can be chosen and used (Duval 2000). Without this internal co-ordination, which must be developed, students cannot step across the threshold of representational conversion. The ability to mobilise various representations together in an interactive way or in a parallel way depends on the development of this co-ordination and conceptual comprehension is not the condition of such co-ordination but arises from its development. In others words what matters first for teaching mathematics is not to choose the best system of representation but to make students able to link quite different ways of representing mathematical contents.

How to make students enter the wide and complex functioning of representtational conversion?

Most teachers agree about the importance of making students use both symbols and pictures, represent spatial and numerical patterns, and identify the same pattern in different contexts and representations. But the main issue is to know what kinds of tasks and activities are available for achieving this goal.

The most obvious idea is to display *several possible* representations *at the same time.* For numbers it can be not only decimal notation, but also fractions or figural representations (polygonal numbers). For functions, both their algebraic expressions and their graphs (visual display of the corresponding lines, curves, surfaces) should be given as if they could be connected like words and images. Software provides powerful tools to display “instantaneously” as many different representations as are needed. Thus students can always get access to a wide range of possible representtations of the objects on which they are working are using as tools. Moreover software can give a dynamic perception of the transformation of representations in contrast to the static ones offered by paper. Specific tasks on transfer or translation can also be given to train students to react to a given kind of representation (verbal, symbolic or visual) by changing it into another one.

In spite of all that, there is a danger of deception because conversion involves two levels of cognitive processes.

— (1) *Identifying the same object* denoted in two representations (from two different registers) *whose contents seem quite different* (an algebraic expression and a graph, a statement and an equation). To recognise this, the activation of an association between two representations of the same mathematical content can be sufficient.

— (2) *Identifying two different objects* denoted in two representations (within the same register) *whose contents seem alike* (between two graphical representations whose contents are visually alike but which do not represent the same functions, or between two statements using the same words but expressing quite different relations or not giving the same information). To recognize this, the activation of an association does not work anymore. In – such a very frequent - situation students must be able to discern the significant elements in the starting representation and the ones from the target representation with which they can be associated. This is the cognitive condition for identifying what is mathematically different within two representations that seem alike, because not all the significant elements within each representation are mathematically relevant. This cognitive condition is particularly strong when the representations are linguistic or visual and not purely symbolic.

Now we can see that only the surface level of representation (1) is taken into account in any simultaneous display or in any task of direct conversion by translation

(Figure 9 below). Thus students are faced with local tasks without links between them, because the only variations which are taken into account are the changes in the mathematical content represented. And the separation between these local tasks is emphasized by the fact that they are set in quite different contexts or problem situations. There is no possibility of identifying and paying attention to the significant semiotic variation which is involved in that change.

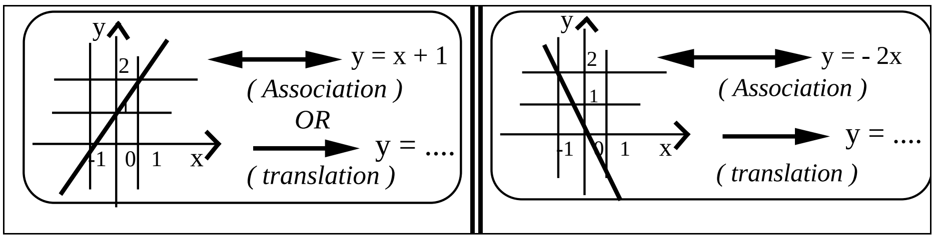


Figure 9. Two local similar tasks confined to the surface level of conversion

Generally, the only way to make the links between two quite different representations of the same mathematical content explicit is to make a commentary or to make a treatment (calculation). In other words the possibility of building bridges between two representations independently of the mathematical content represented is dismissed. Through this common practice it is implicitly assumed that the deep level (2) would be mainly conceptual and non-semiotic. But what is going on when conversion requires a distinction between two similar representations of two different sorts of mathematical content? For example, how can we make students become aware of the contrasting visual features that are both mathematically relevant and visually significant for graphs (Figure 6 above)? The only method for making students able to analyse the specific way a given semiotic system can represent objects of knowledge is to make them compare with the properties of another semiotic system.

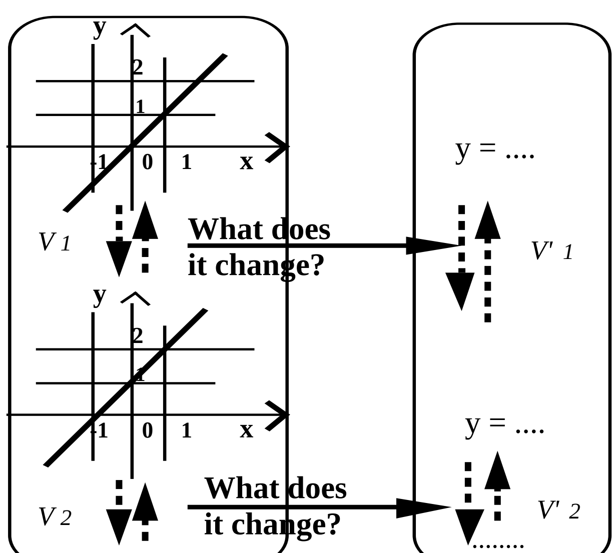


Figure 10. Global task to analyse the concomitant variations in the way of representing underlying any local task of conversion

A learning task aiming at the deep level (2) of representation must meet three criteria with regard to student activity:

— focusing on the variations of representation within the initial register,
— investigating the field of possible variations within the initial register (and for this investigation the basic experimental rule for representational variation in the source register must be observed)
— comparing the meaning of these variations within a register by asking the question: does anything change in the target register and if so what does change?

Such a kind of task is crucial in order to make discriminate whether two representations whose contents seem alike represent two different mathematical object or the same. And that is very often the condition for being able to convert them into another representation register.

Thus we get two different kinds of tasks for making students become sensitive to the variety of possible representations for mathematical objects and relations. In the first kind two representations have to be grasped together by simultaneous display, by translation or by transfer, while the process of conversion is left aside. The association of representations can only be justified by calling for mathematical knowledge, which involves the production of other semiotic representations. In the second kind a field of representational variation is grasped in relation to another field of representational variation. This makes students experiment with the semiotic complexity of the process of conversion and helps them co-ordinate semiotic systems which would otherwise be left in separation. A true conceptual comprehension of mathematics and the ability to make fluent changes of representation are being fostered through this co-ordination

There are as many areas for developing co-ordination of semiotic systems as couples of registers: Language → symbolic notations, language → diragrams, language → figure, graphical representations → algebraic notations of relations, etc. The complexity of this work of learning work obviously depends on both the source register and the target register and therefore on the gap between two quite different ways of representing and working.

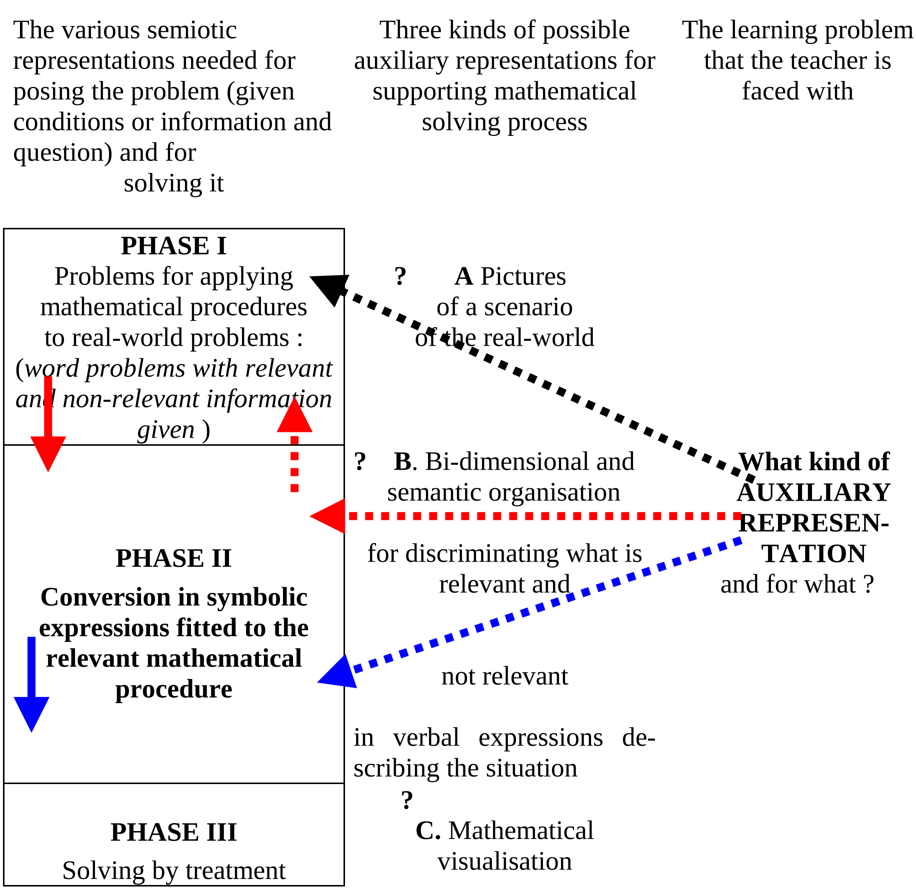
"Real-world" problems and representations

From an educational point of view the paramount problems of daily life are often highlighted, especially for teaching at primary school. There is the common idea that knowing how to apply mathematical operations and procedures to pratical problem situations gives meaning to mathematics learning. There is also another reason which is more interesting for our purpose. Solving "real-world" problems would mainly call for students' physical or everyday experience and their mental representations. Thus students could be spared the trouble that semiotic representations give rise to and, moreover, they could be led to the comprehension of mathematical concepts and therefore they could make sense of the semiotic representations used. In other words, our previous analyses of mathematical thinking are relevant for this aspect of mathematical activity which is claimed important for mathematics teaching. Or, the other way around, does any application of mathematical operations for solving real-world problem require a preliminary articulation of various representations, including semiotic representations?

All situations in which counting and quantifying are a part of a real activity give rises to problems in which the application of number knowledge is required. The most common way to pose a problem taken from such situations is to make students identify and carry out the correct arithmetical operations. One-step arithmetic word problems involving either the choice between addition and subtraction or the choice between multiplication and division are well known. They can also be used for introducing new kinds of numbers (decimals, fractions) or proportional reasoning. A broad use of real problems is also made in order to support the transition from arithmetic to algebra. Thus we get a wide range of real-world problems that vary with regard to situation, context and mathematical procedures.

From an educational point of view these problems give the opportunity to call for a large range of non-verbal representations either connected with concrete experiences or with mathematical operations to be carried out. Faced with these problems students and teachers could choose the best representations for working with the mathematical content involved and for solving the problems.

However, from a cognitive point of view, all these problems involve the same complex processes and hence raise the issue about the relevance of the representations used. Whatever the problem posed, the whole cognitive situation in which learners and teachers are put is the following:



(transformation of representations within the same register)	for understanding the procedure (numerical lines,. diagrams,..)

Figure 11. The labyrinth of representations underlying the solution of “real problems”

Even when quickly forgotten, words are needed at least to describe or evoke a scenario of the real world and for posing a question (first column in Figure 11). Then conversion is needed and is intrinsically connected to the identification of the relevant information for the choice of appropriate arithmetical operations, or for the choice of the unknown quantity to be denoted with a letter. Treatments depend first on this identification and on the choices made, even if different procedures can afterwards be chosen according to the mathematical knowledge at issue.

The educational advantage of real-world problems is to allow for working freely with representations that seem more accessible than the ones used in mathematics (second column in Figure 11). In this way they are like auxiliary representations which can help students’ understanding at each stage of the solution process. But here lies the crucial issue. An auxiliary representation can only fit one of the three phases of the “real-world” problem solving process. Thus iconic representations can fit only with the scenario evoked and can be disturbing for the phases II and III which require non-iconic representations that are necessarily semiotic. Moreover, the representations which can help students understand the way to convert given information are quite different from the ones which can help them understand the way to carry out the mathematical operations. Auxiliary representations can fulfil only a specific function in problem solving, a function which is relative either to imagination, to conversion or to treatment. Anyway, what matters is not to find the “right” representation but a variety of relevant ones and to co-ordinate them.

It would be easy to show that in most research, these cognitive conditions for helpful use of auxiliary representations is entirely ignored. Either the iconic representations or mathematical visualisation for understanding procedures are taken into account, as if they were sufficient for the phase of conversion. There is not a shred of evidence to support this educational hypothesis. Quite the contrary, even most 20 year old students, and sometimes teachers at primary schools, cannot get out of the labyrinth of representations, even for the one-step additive word problems!

Conclusion

The recurrent and systematic difficulties encountered by most students in learning mathematics lead us to ask the question: are the thinking processes the same in mathematics as in the other areas of knowledge? Since Piaget’s theory of epistemological development it is more or less assumed that the cognitive processes are basically common to all areas of knowledge. And research in mathematics education is mainly concerned with the ways in which each particular concept, and each particular mathematical topic set in the standards of a curriculum, can be taught. Even if the obvious need of various semiotic systems is recognised, the basic role of these systems in the thinking processes, and the specific problems they gave rise to in mathematics learning, are therefore neglected. Such a cognitive framework underlying most research in mathematical education comes up against a crucial issue resulting from the cognitive paradox of mathematics: the inability of most students to change representation register. However, the way signs and semiotic systems play an important part in mathematical thinking is not easy to isolate from the denoted objects. Nor is it easy to analyse its cognitive functioning, because every semiotic representation is a representation of something. How can we investigate the basic role of semiotic representations in mathematical thinking and learning?

I have put forward three main ideas. Firstly, what matters in semiotic representations is not their relation to something else, their denoted objects, but their intrinsic capacity to be transformed into other semiotic representations. This is the basic part they play in thinking processes. Each semiotic system provides a specific capacity of transformation. Secondly, there are two kinds of transformations of any semiotic representation: conversion and treatment. They are cognitively quite independent from each other even though in mathematical terms the former depends on the latter. This is the reason why conversion of representation is the first threshold of comprehension in learning mathematics. Thirdly, and that is the most sensitive point, what is concerned with conversion and what is concerned with treatment must be separated when analysing what the students do when they are faced with a problem. Such a methodological and theoretical separation goes against the current practice of considering these two kinds of transformation as making up a whole for problem solving.

In this paper I have emphasised conversion because it is only in mathematics that such a wide and complex game of representational transformations is required for thinking but also because a dualist approach to mathematical activity leads to the denigration of its cognitive importance. In fact conceptual comprehension arises from the co-ordination of the various semiotic systems used, and becoming aware of the specific way of representation characteristic of each semiotic system is a cognitive condition for comprehension. By focusing on conversion it seems that semiotic representations cannot be put on the same level as the auxiliary representations whose use is mainly for individuals or for educational aims. However some treatments have also a specific cognitive complexity, mainly those which use language and visualisation. Thus the way of expressing and understanding linguistic statements is not the same within and outside mathematics (Duval, 2003). There are also quite different and conflicting ways of looking at figures in geometry. They provide ways of functioning which are not under the control of geometrical concepts, which is the reason why they can be heuristically powerful. It is only by separating conversion and treatment that the cognitive complexity of all the discursive and visual treatments which cannot be fixed in algorithms can be described and investigated.

When focused only on the particular mathematical content to be taught, the more global process of transformations of representations which is needed for mathematical activity is left in the dark. And nobody shall be able to answer the predominant question of most people who don’t teach mathematics: How can mathematics learning contribute to general education for shaping the mind and developing more global abilities of visualization, reasoning, information organization, rather than just providing some technical procedures of calculation? This is the reason why analysing the cognitive processes involved in mathematics thinking requires a shift of orientation in the way tasks and problems are chosen for students’ learning and also for research about learning. The cognitive variables relative to the various kinds of representations must be taken into account. This also requires new methods to go beyond what is recorded at the scale of the everyday work in the classroom.

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