Does the school of the 21st century need geometry?

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Neither thirty years nor thirty centuries influence
the clarity and the beauty of the truths of Geometry.
Lewis Carroll
Let no one unversed in Geometry enter my doors
(The inscription on the entrance to the Academy of Plato)

Instead of an introduction

Just 10-15 years ago an intention to write an article with a title such as this one would seem at least strange. Why do we need to defend evident truths? Geometry has existed for more than two thousand years and manages somehow without our protection. What is actually the problem? Especially, if we talk about the Russian school, where geometry is traditionally presented at a very high level, and where the position of classical geometry seems to be unshakable. The number of geometry lessons is shortened a bit, some topics are removed from the program (nowadays it is called “standards”). The general level of knowledge in geometry is slowly decreasing year by year, but this is rather an objective process, going on almost everywhere, all over the world. And this does not only happen to geometry. What are the reasons for panic?

We agree, completely agree. The only thing that remains unexplained is why the question itself “Is it necessary to learn geometry?” being absolutely ridiculous not so long ago, now sounds quite serious. Nowadays this question can hardly surprise anyone, it is discussed on various levels of the educational community. So, what happened? Has the critical point, beyond which quantity affects quality, already been passed, unnoticed as always? Today we hear clear voices demanding the omission of geometry from the school curriculum, or, at least, a considerable reduction of it. Often these voices are heard from people, who count themselves (sometimes, by a misunderstanding) as members of the professional mathematical community. For some strange reasons the most significant shortening was made to courses of stereometry, which is the most difficult part of geometry, but on the other hand, also the part that is, today. the most important one I practical terms. Stereometry is completely removed from the International Mathematical Olympiads! As for the situation in Russian schools, geometry formally remains in the high school program, but actually it is almost reduced to a secondary subject. Having become familiar with the program of graduate testing we have to omit this “almost”’ from our statement. In general, we believe, the testing system is incompatible with geometry. Geometry is not a collection of definitions and formulas, but the ability to see, imagine, and think. It requires a careful penetrating analysis, not one-blow answers.

One of the main reasons for all this, from our point of view, is that classical geometry is no longer respected as a serious science and as school discipline. It is seen as something archaic, which is not up-to-date. Why are we still torturing children with memorizing axioms and with puzzling their brains over difficult problems? What is

[1] Most sadly, Igor Sharygin passed away in March 2004, less than three months before the Congress

the use of being aware that the three altitudes of a triangle intersect at one point and that their bases lie on the same circle that contains the three midpoints of the sides?

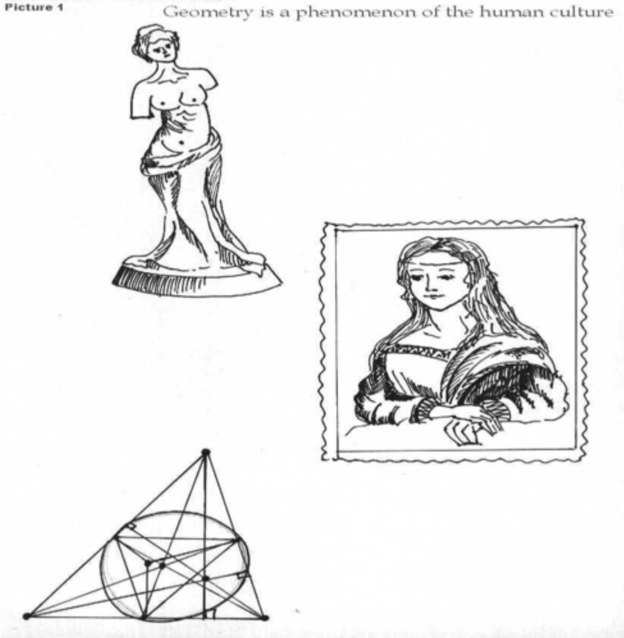
Would it be better to take from geometry only those facts and methods needed for future education, such as the Pythagorean theorem, the sine and cosine laws, trigonometry and vectors? The remainder can hardly help in modern life. Let’s appreciate the achievements of Thales and Pythagoras and then forget about them, concentrating on more vital subjects: informatics, economics, law. Watching what is going on now in and around the educational system, we can hardly get rid of the impression that the old gentle geometry is about to be dismantled, dissembled into bricks. Like marvellous classical buildings, which slowly disappear nowadays from the historical districts of Russian towns being quietly demolished. The free space is immediately occupied by new buildings, in style and “in modern spirit” tasteless and ugly

The main problem is that all of us studied geometry in normal schools using good textbooks. For us this is something very normal; most likely we are not able to realize what kind of people we would have grown up to be without studying geometry. Now such a prospect in the future may come to a whole generation of children. In this article we try to show that geometry is something much broader than just a branch of mathematics or a school subject. It has a strong influence on the development of the personality and on the upbringing of children. Possibly, the consequences of reducing programs of geometry in school may turn out to be catastrophic. We shall also discuss what sort of geometry should be present in school, what, in our opinion, are the aims of studying geometry, and how these can be achieved.

Geometry as a phenomenon of the human culture

One aim of studying geometry is, of course, to obtain knowledge. However, we must admit, this aim is becoming a minor one, for most of the geometric knowledge obtained in school is rather useless both in real life and even in scientific research. What is most important is that geometry is a phenomenon of human culture. Some geometric theorems are older than the Bible, they are masterpieces of the human thought. If someone does not know, say, who painted “Mona Lisa”, or where the Coliseum is, may he or she be considered as a well-educated person? But, what can be said about a person who cannot formulate the Pythagorean theorem or does not know what the problem of squaring the circle is about? Human culture is pierced with geometry everywhere, from Leonardo’s paintings to poems of Joseph Brodsky. In our opinion the knowledge of basic facts from elementary geometry is necessary for any cultured person, in the same degree as is the case with knowledge of history, literature and foreign languages. A person, ignorant of geometry, is lacking in cultural development.

Picture 1 Geometry is a phenomenon of the human culture



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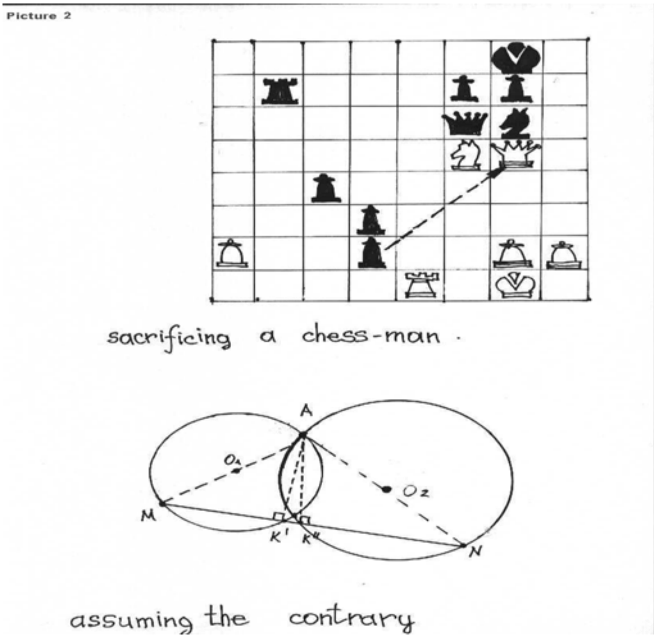
Upbringing by geometry

Geometry, as well as mathematics in general, helps in the moral and ethic development of children. Lev Tolstoy, when characterizing the older prince Bolkonsky in his novel *War and Peace*, wrote: “He said there are only two reasons for human vices: idleness and superstition, and there are only two virtues: activity and intellect. He was bringing up his daughter on his own. In order to develop both of these main virtues he gave lessons of algebra and geometry to her and tried to fill all her time with continuous study.” It is hardly possible to overestimate the moral and ethic aspects of the study of geometry. It may seem strange, but geometry is also a good moral training for a child. Learning mathematics builds up our virtues, sharpens our sense of justice and dignity, strengthens our innate honesty and our principles. The idea of proof is one of the most high-moral ideas in the world. The provability principle is an essence of both scientific and moral aspects of course(s) of geometry. This is the only discipline at school, including even other mathematical courses, entirely based on the principle of the consecutively deriving of all statements which are not axioms. In fact, geometry is the most honest subject at school. Either one has proved a theorem, or one has not, there is nothing in between. Neither nice words, nor broad erudition can help if you don't have a rigorous line of reasoning. An excellent speaker or a well-read erudite pupil has no advantages in geometry lessons at school, as is sometimes the case in, say, lessons of history or literature. Geometry tests the brain capacity and industry of pupils in a most honest way. Geometry also stimulates the courage of a child, the ability to take a risk, to make a brave and unusual decision. A famous mathematician said that when you are assuming the contrary (*reductio ad absurdum*) in solving a geometrical problem, you are making a bolder act than a chess player sacrificing a chess piece. Indeed, a chess player gives away only one piece, but a mathematician gives away the whole game. It is extremely difficult to cheat and to manipulate people familiar with the idea of proof. Let us remark that a state never cares to justify or to prove its actions. Therefore, here’s an advice to those who want to become politicians: never study geometry!



Training for higher education

Geometry is one of the best grounds of training to prepare high school students for universities and technical institutes. This is not restricted to entrance exams only. First of all, students realize the ideas of mathematical proof and rigorous logical reasoning. Geometry develops mathematical intuition, accustoms children to independent mathematical creativity. Thus, geometry, to an even higher extent than algebra, prepares the basis for studying higher mathematics and physics. The fundamental notions of algebra and analysis that children learn in school, will be revisited in university all over again on a deeper level, beginning with the very basic notions – the conception of number, of set, etc. If a pupil has missed something in, say, the definition of the derivative, this would not be a disaster for him or her, for this would be taught again in university. On the other hand, (s)he will never come back to elementary geometry. This subject is completely missed in undergraduate courses. So, school is a unique opportunity for children to study it. Anything missed in the school stage will remain missed forever, which will unavoidably cause problems for subsequent studies. Elementary geometry is not a thing in itself! As a matter of fact, it has a straightforward connection with modern mathematics. The mathematical topics in university (Fourier series, complex analysis, functional spaces, etc.) are actually very geometrical; they are full of various geometric ideas.

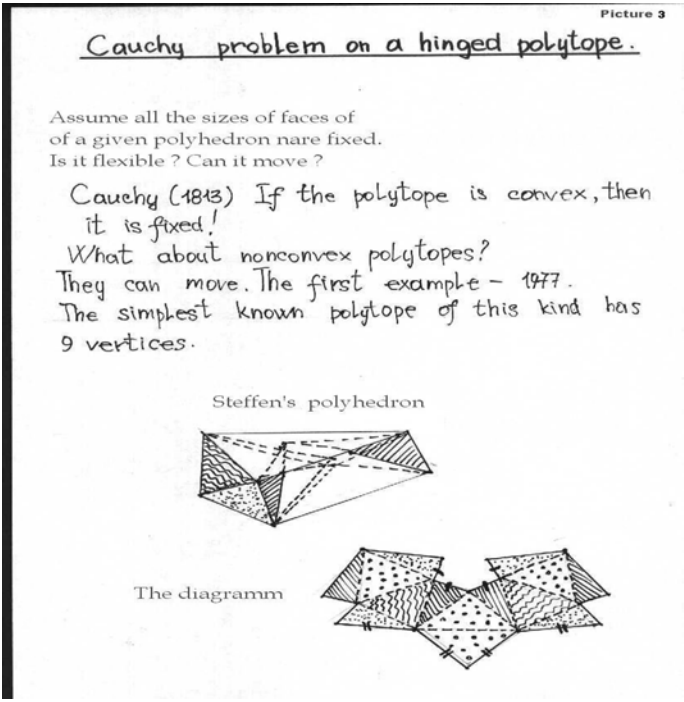
On the other hand, geometrical lessons in school provide a unique opportunity for children to do their own small pieces of research, to make the first discoveries in their life. In this sense geometry is the only area of mathematics, where a gifted pupil can be in competition with a venerable professor and can try himself in a serious science.

Geometry is a minimum point of the distance between school mathematics and high level modern mathematics. Many so-called “elementary” problems in geometry, whose statements can be easily understood by school students are still unsolved, while others have been solved just recently. At first sight, the situation is similar to that in number theory. However, the main difference is that (and this is most astonishing!) in geometry the solutions of these difficult problems are often quite as elementary as the statements. In most cases they can easily be explained at the school level. For instance, let us mention the famous example of C. Steffen of a polyhedron (a nonconvex hinged polyhedron that can move, having all sizes of its faces fixed) in a regular textbook of geometry for Russian schools. Let us remember that the existence of such flexible polyhedrons with fixed faces used to be a great problem formulated by the famous Cauchy in 1813 and solved relatively recently, in 1977, by R. Connelly. Children are provided with a complete description of this polyhedron to make it with paper and glue on their own. What can be more exciting and encouraging for a future researcher than to touch with your own hands a more than 160 years old problem and to realize that there is nothing supernatural in it!

Many other newly obtained achievements of mathematics can be brought directly to school lessons, and in our opinion, this may happen only in geometry, neither in algebra nor in physics. Sometimes students solve such problems themselves. Let us remind the reader of just two examples of this kind.

The first one is an elementary example of a convex polyhedron whose flat diagram has overlaps (the existence of such polyhedrons also used to be a problem). It was constructed in 1997 by a first grade student S. Tarasov. The example is surprisingly simple. This is nothing else than a frustum of a regular triangular pyramid. Moreover, shortly afterwards this example was taken directly to a Mathematics Olympiad for high school students. The problem was to compute the angle at the vertex of the pyramid with overlapping diagram. The problem was considered there as a rather simple (!) one (that problem was given as number 2 out of 6 problems, which were enumerated according to growing difficulty). By the way, the answer is rather curious: the angle is at least 100^0. Such a value might be natural in some physical problems, but not in a geometrical one!

The second example concerns a well-known conjecture of H. Poincaré on three closed geodesic curves on a smooth convex surface. In other words, for any piece of stone with a smooth surface (say, a pebble) there are at least three different ways to gird it with a rubber circle so that the circle will not slip off. The conjecture was proved by L.A. Lusternik and L.G. Shnirelman in the late 30’s. To everybody, including H. Poincaré and D. Hilbert, it was more or less obvious that in general the number of geodesic curves cannot be increased, because for an ellipsoid (with unequal axes) there exist exactly three such geodesic curves. D. Hilbert and S. Cohn-Vossen even wrote in their book *Geometry and the Imagination* that an ellipsoid clearly possesses only three geodesic curves, and that the proof is very simple. However, in the 80’s it was shown that any smooth convex surface possesses infinitely many such curves. But what about an ellipsoid? Does it have many geodesics as well? Were Hilbert and Poincaré both mistaken? Recently in our geometry seminar in the Moscow Independent University we came across this theme. The question arose is there a fourth geodesic curve on an ellipsoid? Is it possible to grid an ellipsoid with a rubber circle somehow in a different way, than by putting it along one of the three equators? Everyone was sure that (if it is possible at all) it must be very complicated, since neither Hilbert nor Poincaré found it. However, a week later two young guys (one of them attends our seminar) came up with an example of the fourth rubber circle. The example is extremely simple and obvious, it can be explained in one minute not only to high school students, but even in primary school! It is a kind of miracle that great mathematicians did not notice this! Thus, we see that sometimes usual geometrical intuition can do more than powerful tools of modern mathematics.



Aesthetic development

An outstanding Russian historian A.C. Kozarzhevsky once told to his students in a lecture: “The aim of my course is to teach you to distinguish real beauty from decoration. This is not so simple as it seems. An aesthetic taste is needed for that. Beauty is not curls and shells on the façade of Morozov’s house in Vozdvizhenka street, beauty is the rigor of proportions and the perfection of forms in the classical Pashkov’s house”. How to develop the aesthetic taste and the sense of beauty of our children? In the lessons of history or arts students are mostly rather passive listeners than creators. Development of the creative talents of students is traditionally one of the central goals of the Russian educational system. Nowadays, this goal turns out to be especially pressing. Unfortunately many kinds of traditional creativity, which used to be popular in the past, such as handicrafts, sewing, painting etc., are becoming less widespread among children now. Studying geometry, solving geometrical problems, as a kind of “intellectual crafts”, can preserve the creative potential of children. Geometry evolved not just out of the practical needs of human beings, but also, we stress it, out of their spiritual needs. In this respect, it can be compared to poetry, music and painting. Mathematics in general, especially geometry, possesses a very special sort of aesthetics. It is impossible to understand the complete significance of geometry, if one doesn't see the beauty of geometric forms, formulae and statements.

Geometry and the history of human thought

The history of geometry reflects the history of human thought. If we carefully trace the development of geometry back from ancient Egypt and Greece through medieval times and the renaissance to the modern history, we can follow the development of science and the transformation of human ideas about the world and the universe. The geometric constructions of the Ancient Greeks were related to their contemplative philosophy and the faith in the “World Harmony” of Pythagoras. In analogy, the development of mathematical knowledge in the renaissance reflects general deterministic principles, the conception of provability, and of rational arrangement of the world. In this respect, some modern notions of geometry, such as fractals, attractors, self-similarity, irregular curves and surfaces etc., reflect the peculiarity of modern human life. The world around us becomes “nonlinear”, less regular and less stable. In the beginning of the twentieth century irregular objects in mathematics were considered to be somewhat “pathological” (let us recall the words of A.A. Markov: “All continuous functions are differentiable except for those artificially constructed by mathematicians”). Nowadays such pathological creatures are becoming one of the main objects and tools of mathematics. Thus, *the development of human thought and of ideas about the world around us is closely connected with the development of basic geometric ideas.* Galilei said: “The book of nature is written with triangles, circles and other geometrical figures. Without them a man could not understand a word in that book”. It is hardly possible to overestimate the crucial role of geometry in the development of other fields of modern mathematics and other sciences. The contemporary notion of a number (due to Weierstrass and Dedekind) as well as the fundamental conceptions of analysis (completion, Dedekind sections, the Archimedean principle) actually came from the intuitive human concept of the geometry of a straight line. The attempts of Pythagoras to compute the length of the diagonal of a unit square gave birth to the modern notion of a number. Who knows, maybe, if the square root of two did not appear, to the Ancient Greeks, as such an obviously important object in geometry, the human species would still use only rational numbers, and the all of mathematics would have developed in a different, entirely constructive way. In that case mathematicians would still manage without existence theorems and precise calculations. The old problems of the doubling of a cube and of squaring the circle eventually gave rise to Galois theory and the notion of transcendent numbers. Modern optimization theory comes from the barrels of Kepler and Fermat's problem of a rectangle of the maximal area with a given perimeter. This list of examples can be continued, but let us stress that the influence of geometry on the whole of mathematics is not a matter of the distant past; geometry still has many resources allowing it to supply other sciences with new ideas. Let us give two short examples. One is wavelets and frames, modern tools in signal processing and computer technology, where they are successfully replacing classical Fourier series. As a matter of fact, the construction of both wavelets and frames is very geometrical. Another one is the interior point method, the most powerful tool in modern optimization. It is not only very geometrical, but has also provided new notions to elementary geometry itself (we are referring to the so-called analytical center of a figure). Thus, the development of geometry is really needed for the progress of other sciences.

What sort of geometry?

Now it seems to be clear that geometry is one of the most important subjects not only among the mathematical sciences, but in the school curriculum in general. The potential of this subject embraces an incredibly vast area of applications. It can be related to all known purposes of education. Why, then, does the role of geometry in school inspire so acute arguments?

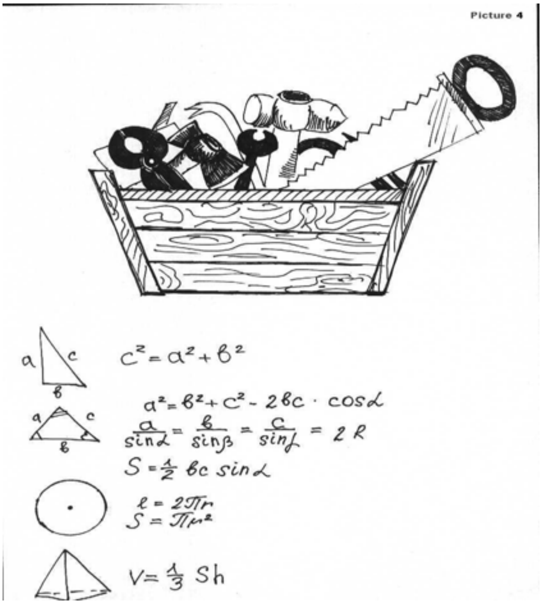
One should observe that the discussion concerning school geometry resembles very much the well-known argument from "Through the looking glass" of Lewis Carroll: "You might make a joke of that: 'horse' and 'hoarse'. Two similar words denote two totally different things." Thus, before we address the question(s): "Shall geometry remain in the 21st century school curriculum? Why shall it remain there?" We must explain what kind(s) of geometry we are talking about. Geometries vary.

Rephrasing the well-known saying of Tolstoy's, one can tell: "Good courses of geometry can be organized in many different ways, while bad ones are similar to one another." There exist three principal ways to rule out true geometry from the school, and so there are three main types of anti- (pseudo-, quasi-) geometrical textbooks. Moreover, in spite of the manifest differences between their basic approaches, textbooks of all these three types are similarly badly structured, poorly written and illustrated, and full of logical flaws.

It's amazing that logical gaps and mistakes are characteristic of the textbooks pretending to be written with full compliance with the laws of formal logic and the *axiomatic method*. Textbooks of this sort are quite widely spread, in particular, in Russian schools. One can distinguish them by the following features: plenty of formal definitions, which make the defined object barely recognizable; unreasonably much space devoted to the basic notions, resulting in students learning nothing new for at least half of the course; too much verbose reasoning, or rather void combinations of words posed as reasoning, used to prove completely evident facts while actually making those facts absolutely vague and, which is still worse, discrediting the very idea of proof. For instance one modern geometry textbook contains the following problem: "Prove, that a segment has only one midpoint" and many other problems of this sort. Such textbooks are sure to very soon kill children's interest in the subject. They present geometry as a senseless demagogical game, the main purpose of which is to compose sentences of a special type and to pronounce them solemnly.

Excessive formalization of geometry is a rather old phenomenon. For instance, this is the principal defect of A.V. Pogorelov's textbook (not at all the worst textbook in geometry in modern Russia), one of the main geometry textbooks in Russian school for already more than 20 years. A lot of space is taken up by such evident statements as "show that any isometry of a plane maps a line segment into a line segment, a circle into a circle and a triangle into a triangle." We used to witness an intensive discussion at a mathematics teachers' conference, caused by the question, how one should understand the phrase "a triangle having a given position with respect to a given half-line" in that book. However, even though the share of formal logic in A.V. Pogorelov's textbook is unreasonably big, its reasonings are not void. Formal reasoning is not the goal of this textbook but appears as an inevitable byproduct of the attempt to give all foundations and proofs within the frame of school geometry. Textbooks, in which rigorous formal reasoning replaces geometry itself instead of helping one to understand it, appeared later. In fact, they are "achievements" of the present time.

Another kind of pseudo-geometric textbooks are the so-called "*practical courses*". Here practical purposes are reduced solely to every-day problems. The contents of these courses are narrowed down to a short list of formulae for calculating lengths, areas and volumes. Textbooks of this sort prevailed in Russia at the dawn of the Soviet era, and now they are typical in many Western schools, in particular in American schools (as far as we know). Historically such textbooks can be justified by the etymology of the word "geometry". However, modern geometry as a science has already for a long time exceeded the narrow scope of "land-measuring". Nowadays human every-day activities pose various practical problems including geometrical ones that lie very far from the "land-measuring". *So, it turns out that courses of both types considered so far fail to meet the objectives they declare: logical ones abound in logical flaws and practical ones give little knowledge that is applicable in practice.*



While with these two kinds of geometry textbooks everything seems to be clear, the situation around the third sort of anti-geometrical courses is not so obvious. Here we mean the "royal route in geometry", established by Descartes. The *coordinate method* discovered by him allows, as he believed himself, an average and even a mediocre person to reach the level previously accessible only to a gifted individual. A later classic observed that Descartes "covered geometry with the mange of algebraic formulae". One should admit that coordinates give a uniform method for solving most complicated geometrical problems. For some geometrical problems this method is the most natural and the easiest one. For example, consider the following problem: given two circles, find the set of points in the plane such that the ratio between two segments of the tangents drawn from this point to these circles (from this point to the points of tangency) equals a given value. The shortest and most natural solution of this problem is by the coordinate methods. There are many problems of this kind in stereometry. For instance, given a unit cube $ABCDA_1B_1C_1D_1$, find the radius of a sphere passing through the vertices A and C_1 and through the midpoints of the edges AA_1 and BB_1. So, one shouldn't get rid of the coordinate method at all. This method is very powerful and helpful, but it is only one of many methods in geometry, and it should not be considered as the main or universal one, on which the whole course of geometry is to be based. A textbook in geometry should contain a carefully chosen collection of problems for the coordinate method to develop skills in this method and, which is even more important, to teach children how to recognize problems where this method is best (many right angles, equations of all acting figures can be easily written, etc.)

The coordinate method cannot replace geometry! Even among the winners of International Mathematical Olympiads there are students that wield only the coordinate method, but their skill in this field is so great that they are able to solve virtually every geometrical problem proposed at the Olympiads solely with this method. However, we are convinced that the coordinates method (as well as trigonometry) is one of the most efficient ways to fight geometry and even to eliminate/kill it. In spite of this its popularity, even among gifted students, is growing rapidly these years. This is a very threatening tendency! Some 5-10 years ago the organizers tried to fight against it, though (we testify!) their task was really hard: they tried to select problems with nice geometric solutions which would have been hard to solve by calculation. Nowadays, it seems that everybody has got used to the fact that many (if not most) Olympiads contesters would not even try to look for your beautiful solutions, but will resort to a “cheap, efficient and practical” method: to introduce a dozen of parameters, to cover the plane with affine or barycentrical coordinates and finally “crack” your problem without much aesthetics. We are not only talking about gifted students. Even though “breeding” of a mathematical elite is a very important goal, but as a poet said: “In a real tragedy it is not a hero who perishes, it is the chorus”. Algebraization of geometry is harmful at all levels of secondary education, equally dangerous for weak and for talented students. As far as weaker students are concerned, the threat to them from extreme algebraization is evident enough. Most students in this group have problems with counting, have difficulties with understanding and memorizing formulae, etc. Rather often it is not their fault. As a matter of fact, logical thinking is governed by the left hemisphere of the human brain, while the right one is in charge of our aesthetical activities. It has been shown that it depends only on the individual, which of his/her hemispheres is dominating. We argue, that for “intellectual left-handers” geometry could become a remedy subject to increase their status in the class, compensating for drawbacks in their mathematical maturity. And instead of this, coordinate geometry would add to the heavy load of algebra on their brains, involving them in a dull and, to them, hard activity.

Why is the algebraic-coordinate approach dangerous for gifted students? The crux of the matter here is that using the algebraic (coordinate) method to solve a problem one often misses the geometric meaning of the situation considered. Teaching pupils to solve problems by the coordinate method we risk breeding a soulless executor, trained just to solve a class of given problem. No less, but also no more. Geometrical and, more generally, mathematical intuition, which is an important ingredient of any investigation, is not at all involved in this process. That's probably the (partial) reason why winners of the International Mathematics Olympiads seldom turn into first-class scientists.

Of course, these three types do not exhaust bad geometry. Here and there one comes across various logico-practical mixtures or modernist, and even post-modernist, integrated courses of natural sciences. However, we stress it again, these courses are easily recognizable. It is enough just to look at the table of contents and to browse through the textbook.

Now it is more or less clear, what sort of geometry we consider unacceptable. A natural question then arises: what kind of geometry is desirable? It is hardly possible to give a comprehensive answer for this question. Everybody, whether a veteran or a beginner, has his or her own – quite personal - idea of a perfect textbook. Still, one thing is indisputable. Rephrasing a famous saying by the former leader of USSR L.I. Brezhnev “Economics has to be economical!” (this was the motto of the Soviet economical doctrine for a long time), one can say: “Geometry has to be geometrical!”

not analytical, or algebraic. *Like Algebra, geoetry possesses its own method of investigation.* The main goal of geometrical education is to learn this method.

Here's one more thought about this, which seems absolutely evident to us, but which will probably cause much contradiction. Geometry textbook must not be reduced just to the exposition of theoretical constructions. The process of learning geometry must include various types of intellectual activity. One such type of activity, and probably the most important one, is problem solving. Problem solving is not just a matter of skills, it is a matter of knowledge as well. A student has to become familiar with a set of rather difficult geometrical problems, master several different geometrical methods, and has to learn how to solve problems in accordance with given examples. By the way, the process of teaching algebra consists of similar components: we show different methods and approaches to a student, explain algorithms that are hard, if possible at all, to find independently. However, unlike in algebra, in geometry there are only a few such algorithms, almost none. We would rather say, there are no universal algorithms at all. Still, like in all handicrafts, there are plenty of particular methods, approaches and ideas. Solving a geometrical problem is not just carrying out an algorithm, it is a sort of fine art. Almost all geometrical problems are non-standard. Therefore, the role of basic problems, that is, of the problems whose main purpose is to acquaint the student with a useful fact or to illustrate a method/approach is especially important in geometry. One must not give students problems detecting only the lowest level, “C-level problems”. A problem should be a full-fledged problem, and we must evaluate the degree to which the student has moved from null to the full solution.

Teaching geometry to gifted students lagging behind

The most important socio-pedagogical problem of the modern educational system is the differentiated education, i.e. teaching students with different levels of intellectual development and abilities. The role of geometry in this process is crucial. Geometry is one of the rare tools, probably even the only universal one, working at all stages and levels of education, including the extreme or even especially in the extreme cases: when one teaches gifted students and when one teaches weak ones.

One should always keep in mind that the intersection of these sets (gifted students and weak students) is not empty. It is the former class of students, the gifted ones, with whom most of us (here we do not mean school-teachers, but professional mathematicians-researchers involved in mathematics education) usually work. However, one should remember that among presumably weak children there are many victims of an unbalanced curriculum, inadequate teaching methods or even of teachers’ prejudices. Here we encounter the most important socio-pedagogical problem: how to help such children to get rid of the label “weak”. It is quite possible that a presumably weak student would once turn out to be an extraordinary talent. Just recall the names of such geniuses as Newton, Einstein, Poisson, etc., who were treated as untalented or even lagging behind in school. This list is, of course, just the tip of an iceberg; the phenomenon occurs rather frequently. However, the social value of the rehabilitation of really weak students is even more important.

In some sense, for gifted students geometry is a sort of sport, in which there are professionals, champions and even record breakers, but for weak students it is like intellectual PE classes.

The ways in which geometry manifests its ability to correct student's development and to accelerate it varies with the different stages of education. (Here we consider an ideal educational process in school, i.e. the school as it should look, not as it actually does look. As a poet said: “The purpose of our life is not in what we have, but in what we believe it should consist of”). In primary school and in the beginning of secondary one (children from 6 to 12 years old) geometry is but a sort of PE lessons, an intellectual PE lesson. A student can join a geometry class at any time. One should admit that this is not typical of mathematics. It is usual in mathematics lessons that even a short absence, and ensuing ignorance or misunderstanding of only one topic, can lead to a serious lagging behind that is hard to do away with.

It is a tradition in the Russian school (and we do not see any reason to give this tradition up), that it is in the 7th grade that the systematic study of geometry starts. It is at this stage where the liberal attitude to students, so natural at the earlier stages, disappears. The course is constructed in strict accordance with a given sequence of topics (depending on the textbook chosen) that would be completely broken if just one link in the chain were omitted. (By the way, it is at this stage that the course of algebra is usually divided into a few separate sub-topics) What should be done to make the systematic study accessible to various categories of students? In our opinion one should first of all pay special attention to the beginning part of the course (7th grade). Let's point out two principal tasks, specific for this stage: awake the interest of the student and teach him or her. It would be our luck, if the student has met good geometry earlier, in primary school. If he or she hasn't, then our main purpose is to arouse his or her interest. And this task is in a manifest contradiction with the need of “systematic exposition”, which means that we should look at a student (as far as his knowledge of geometry is concerned) as at a sort of “tabula rasa” - clean sheet of paper that we must fill with various geometric facts ordered in a certain sequence. We cannot use the facts he or she already knows, base our exposition on the knowledge that he or she has contracted somewhere outside the class, we cannot even appeal to his or her ‘common sense’. At this stage we undergo a serious danger of killing any interest in the subject in the student, inculcating in him/her an idiosyncratic attitude towards the subject, nothing sais about being interesting. However, the goal of “awaking interest” at this stage can be reached. Principal tools for this are nice drawings/figures, good problems and live language. We should keep the door into geometry open for a long time, trying to lure students into the subject. We should try to develop a sort of addiction, intellectual and psychological, to geometry in the student. If we succeed, it would be easier for us to achieve the main goal of the second stage of the systematic course, i.e. to teach the student. And at the third, final, stage of secondary teaching, we meet a new important methodological problem, the problem of repeating. At this stage we have an opportunity to fill all the gaps and lacunae in the course, and (which is most important) to show to the student geometry as a single unified subject, a beautiful and orderly building that we have been constructing together for three years.

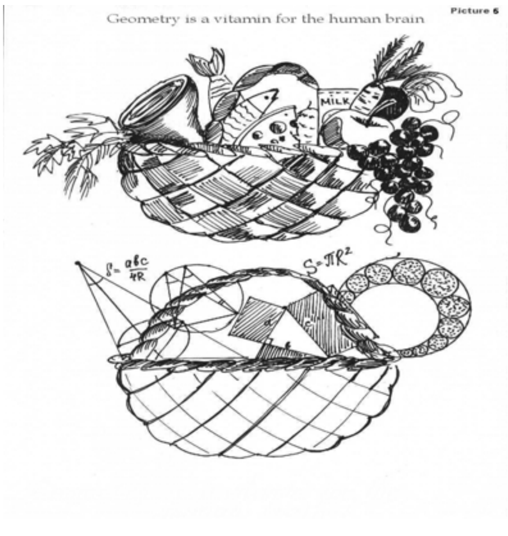
Geometry is very important for the full-fledged physiological (not just intellectual) development of a student. The very process of studying geometry has a great significance in this respect.

Geometry is a basic sort of intellectual activity of mankind
as well as of the individual.

Human science started with geometry. A child starts to learn the geometric properties of the surrounding world before he or she learns to speak. Although ancient geometers (Archimedes, Apollonius) didn't have an algebraic apparatus at their disposal an algebraic apparatus, their results still amaze modern scientists and make us feel marvel. Carrying on the analogy between humanity and the individual, we can observe that the geometrical achievements of younger students do to only a small degree depend on their level of general mathematical training.

Our work with children should be based on the following dual principle: democracy and elitism.
On the one hand we must give to children of all social strata equal opportunities to obtain high-quality education, and we must keep the doors wide-open for the talented youth at all stages of the educational process (which is democracy). And on the other hand, we should provide a higher level of training for the gifted students, in order to create a real intellectual and scientific elite. Talent and good training are the two criteria that should govern the forming of this elite. Geometry perfectly fits the principle formulated above.

When working with gifted students *we must observe yet another principle: comfort and many stages.* The system of Mathematical Olympiads, which is an important element in the training of gifted students, combined with the educational process, often results in a very unfriendly and highly competitive environment for children. This environment can injure the psyche of an unprepared child very badly. Remember that gifted children usually are especially vulnerable. Therefore, it is very important to determine for each student the difficulty level, at which can work comfortably. A too high level can prove to be unbearable for a student. As a result the child can loose his or her self-confidence. At the same time if he or she wastes too much time at a too low a level, this could cause a lag in his/her mathematical growth, and even make it stop. The multi-stagedness and hierarchy manifest themselves especially distinctly in the Olympiads system. The gap between the International and the school Mathematical Olympiads is enormous. Geometry is the back-bone subject unifying this multi-stage system. In addition, geometry allows gifted students to grow up gradually, it prevents them from a too early and forced development, which is an often seen reason for their eventual quitting of mathematics.



Geometry in the 21st century in Russia and abroad. What's new?

Mankind has recently entered a new century. Let's look around, what is going on?

Limits of the human civilization have become considerably wider during the last few decades and go on to grow rapidly. Many educators from different countries undertake desperate attempts to catch up with this growth. Today, one can figure out two principal approaches to this problem: modernization (in a very restricted sense) and differentiation. Often both terms are understood in a very primitive way.

What do the *"modernizers"* of education suggest?

Recently many new professions have appeared, new areas of human activity, even new sciences such as information technologies, etc. Hence we have to restrict the role of old and traditional subjects in the school curriculum and replace these with modern ones. As far as mathematics is concerned, we should first of all cut down the time devoted to geometry (partially, or even completely), since it is an obsolete subject that has changed very slightly during the last few thousand years and has very little practical use. Instead, we should introduce modern branches: calculus, probability theory and others.

What's wrong in this argumentation?

The problem here is that the educational process obeys very strict biological laws. One cannot accelerate it, just like one cannot shorten the time of bearing a child, even though this process includes many stages that are perfectly useless from the point of view of an adult individual. There doesn't exist a magic elevator that can raise a child or even a young person directly to the higher stages of civilization. There have been many attempts to circumvent this law in education, including mathematical education, but these all have failed.

The higher a building, the stronger its foundations should be. A person with a good fundamental education would have much less trouble accommodating to the new realities of life, would much faster find his place in the world than a person having a superficial acquaintance with numerous modern devices, having learnt to press buttons of various complicated gadgets, but knowing nothing about the processes that go on inside these gadgets. The geometric method proves very handy for modern life, since it enables one to get very quickly to the matter of a complex phenomenon, to find a clear interpretation of it.

The *differentiation* of education (in a broad sense modernization includes differentiation) suggests a somewhat different way to solve the problem arising before the modern society. The school, especially the high school, should become highly specialized, there should be a variety of types of schools: humanities oriented, mathematical/physical, biological, even sportive and musical and God knows what else. On one hand that's really necessary. But on the other hand, an excessive differentiation at school level can impede students' realizing, in the future, their basic human rights such as the right of free traveling, the right to choose one's profession etc.. As recent polls showed, a person has to change profession rather often, up to 25 times during his or her life.

In addition, human society is not an ant-hill, where it is possible to grow soldiers or servants, workers or producers by merely changing their nutrition. It is hardly possible to determine the individual's future at the school level. We cannot decide for them. The very intention to do it is totally immoral.

Here we come to the conclusion once again that it is the fundamental training of our pupils that must be intensified. Still, the principles of differentiation must not be abandoned. It is important to determine a reasonable limit beyond which education will break down into separate feudal kingdoms. It seems that from the point of view of mathematics one can do with only two types of specialized school. Without going deeper into details, let's point out that mathematics courses based on visualization principles, on geometric ideas help people whose interests lie far away from mathematics (in the humanities or elsewhere) to obtain a full-fledged (or at least sufficient) mathematical education.

Computers have become a significant symbol of modern civilization. Here we shall say a few words about the relations between geometry and computers. On one hand, the geometrical kind of thinking lends itself to computerization in the least degree. (By the way, from this observation it follows that preservation and development of this type of thinking are of a major importance today.) Geometry remains one of the rare fields of intellectual activity where man is not yet lagging behind computers. And, on the other hand, nowadays the computer provides an efficient tool for research on geometry. One can use it to detect some interesting geometric facts experimentally. The most important task of proving these facts and drawing conclusions (just compare!) is reserved for people. It is remarkable that this experimental activity is equally accessible to strong and weak pupils (as far as mathematics is concerned), people inclined towards the humanities and those aiming for technical applications. Thus, it turns out that *due to the modern computer technologies a pre-science such as geometry has obtained a new impetus for its development, both in scientific and educational respects.*

Nowadays, the rate of changes in our environment is so high that humankind as a biological species hardly has enough time to accommodate.

The influence of ecological movements in society has been considerably increasing recently. People are worried about the quality of the air they breathe, the water they drink, the products they eat. They pay attention to, whether these products are natural or synthetic, whether they contain artificial or genetically modified components, etc. We think it is high time to create an organization to protect the ecology of the educational environment. In fact, we are sure that today a thorough investigation of *the ecology of educational environment* is necessary.

For its normal physiological growth a child requires a well-balanced nutrition. Equally, a well-balanced and plentiful intellectual nutrition is necessary for its normal intellectual growth. Nowadays mathematics and especially geometry is one of the rare ecologically pure and well-balanced products available in education. Geometry can and must become a subject intended to equalize our mental activity, mending the functional disproportion between the hemispheres of the human brain. *Geometry is a vitamin for the human brain.*

However, geometry is a dish, which demands a very well trained cook. Otherwise it can prove not only useless, but even harmful.

One ought to remember that geometry appeared not just from human’s every-day practical needs, but also from our spiritual needs. Still today, it gives everyone an opportunity to reach the world of the pure and ideal. That is what we really want in our “marketized” every-day life. Someone said very well that “Human life is measured not in the number of years we live, but in the number of times we reach the God”. The joy of creation, the joy of victory, the beauty of the world, ``reaching the God'' - all this can be found in geometry. In the world of geometry all conceivable ideals of humanity can be achieved. It is the world of equality and fraternity. In this world people professing all religions in the world, and atheists as well, talk one language, and renowned scientists address novices and peers alike. We know many people, whose profession is far away from mathematics, but whose love for school geometry has survived throughout their lives. “Beauty will save the world,” - said great

Dostoyevsky. Isn't it time to start saving? What about an international program like "geometry against drugs", or "geometry against terror?"

The circle has closed. We mean, it can be closed. Geometry stood near the cradle of human intellect, and once again it can help mankind make once again great advances in its intellectual, moral and spiritual development. We mustn't loose this opportunity.
...
Let no one unversed in geometry enter my doors?..