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The dual nature of mathematics

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Abstract

The vitality of mathematics depends on an inspired symbiosis between the mathematical description of concrete phenomena from the real world and the development of abstract mathematical ideas to explain and understand them. Mathematical modelling is a key component in the symbiosis between concrete and abstract aspects of mathematics, but it goes deeper than that. It is much more like a yin-yang relation where neither aspect can flourish without the other. Accordingly, the teaching of mathematics ought to include both concrete and abstract aspects of mathematics right from the beginning of the educational system. To succeed in this endeaveour is a constant challenge to mathematics educators at all levels from primary school to university. I shall discuss these points of view together with ideas for the presentation of selected elements of mathematical thought.

Introduction

Mathematics originated in the early needs of mankind for counting and measuring in relation to both quantities and spatial objects. A particularly interesting early mathematical artefact is the Ishango bone found in 1960 on the shores of Lake Edward on the border of Uganda and Zaïre. The bone is named after a small settlement living at this location in prehistoric times and it is generally supposed to be about 11000 years old. There is evidence that the Ishango man has carved the bone according to some kind of pattern. The carvings might indicate that some arithmetic was done. Other observations suggest that the bone could have been a lunar calendar, but all this remains speculation (Huylebrouck, 1996). There is evidence of mathematical activities in Africa more than 30000 years ago, and after the Ishango bone was found in 1960, it has generally been accepted that mathematics in ancient Egypt in relation to the pyramids and surveying have an African background. From Egypt mathematics found its way to Mesopotamia, where mathematical source material is known from about 1800 BC.

Neither in the above-mentioned early cultures nor in ancient China, with known mathematical sources from about 300 BC, is there any evidence of systematic formal mathematical theories or proofs of mathematical results. Such activities began with the Greeks around 600 BC; cf. (Kline, 1972). Others may contest that (Joseph, 2000).

With the introduction of formal mathematical theories, abstraction became a useful prerequisite for mathematics and maybe today mathematics can best be understood as a framework for studying concrete real-world phenomena in terms of the underlying abstract mathematical models. Mathematical modelling is a powerful symbiosis between the concrete and the abstract. This dual nature of mathematical

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modelling is both the strength and the weakness of mathematics in the educational system. On the one hand many students find the fascination of mathematics exclusively in the abstract aspects, and on the other hand abstraction is difficult to many students potentially interested in the applied concrete aspects. In mathematical modelling, however, the abstract aspects are inseparably related to the concrete aspects like a yin-yang relation where neither aspect can flourish without the other.

Right from the beginning of the educational system the teaching of mathematics ought to include both concrete and abstract aspects of mathematics. To succeed in this endeavour is a constant challenge for mathematics educators at all levels from primary school to university. In the following we discuss these points of view, together with ideas for the presentation of selected elements of mathematical thought.

A geometric approach to the real numbers

There are many advantages to teaching the real number system by using a geometric approach to visualize the concepts. The real number system is a highly abstract structure, the understanding of which can be eased for visually oriented students by linking the numbers to points on an oriented line - a number axis. It has to be admitted at once that one of the deep basic difficulties in the foundations of mathematics consists in formally linking the real numbers to points on a (mathematical) line, which in itself is a highly abstract construct. A geometric approach offers an intuitive feeling for students of all ages, however, and provides a path to more rapid progress in the early stages without being stuck with deep philosophical and set theoretical questions. And a geometric approach can even help to illuminate the profound nature of the philosophical problems in the foundations of mathematics.

Choose an oriented axis, a line with a preferred sense of direction. The choice of the axis is arbitrary but once chosen, it is kept fixed. Furthermore, we choose a fixed subdivision of the oriented axis into intervals of equal length.

Marking the rational numbers

We can mark the *integers* (whole numbers),

$$\ldots, -2; -1; 0; 1; 2, \ldots$$

along the division points, by choosing one of the division points as 0 and marking the positive integers in the positive direction according to the chosen orientation of the axis, and the negative integers in the opposite direction from 0. The positive integers, called the *natural numbers*, have been employed by humans in an intuitive and non-conceptual manner, even in the oldest cultures; some cultures didn't go beyond 2, though. Much later, the negative integers were introduced by Hindu mathematicians to represent ‘deficits’; the first use of negative numbers is often ascribed to Brahmagupta about 628, but it goes back to about 400 AD. It was also around that time that the Hindus began to use the number ‘zero’ as a usual number; earlier the Egyptians and the Greeks (sources from about 300 BC) had used `zero' only as a `place-holder' to indicate the absence of a number.

If we subdivide each of the intervals of equal length on the oriented axis marked by the integers, in q subintervals of equal length, we get a set of division points along which we can mark all *fractions* with *denominator* q and an arbitrary integer p as *numerator*, the numbers p/q. By letting q run through all the natural numbers we can thereby mark all fractions, representing the so-called *rational numbers,* along the oriented axis.

Catching the real numbers

We now realize that there are points on the axis that have not yet been included: there are ‘holes’ in the axis. For example, the Greeks discovered that the diagonal of the unit square is a quantity that cannot be represented by a rational number. If we lay down this length from 0 we arrive at a new point, $\sqrt{2}$. Before long we realize that there are many more ‘holes’ in the axis than points corresponding to rational numbers: examples include $\sqrt{5}$ and π

We now introduce the *real numbers* as the magnitudes represented by the lengths of intervals with one of the endpoints at 0 and the other endpoint at an arbitrary point on the given oriented axis; the magnitudes are counted with sign corresponding to the orientation of the axis. In this way, the real numbers are identified with the points on the oriented axis, which accordingly is referred to as a *number axis*: in particular, we get $\sqrt{2}$ as above. The real numbers that are not represented by rational magnitudes are called *irrational numbers*.

If we want to describe the real numbers completely from the rational numbers, we can do it most expediently by the following procedure. We imagine that we catch the real numbers in so-called *nested intervals*. By a *nested interval* sequence we understand a decreasing *sequence* of closed intervals

$$[a_1; b_1] \supseteq [a_2; b_2] \supseteq \ldots [a_n; b_n] \supseteq \ldots,$$

in which the length of the interval $[a_n; b_n]$ approaches 0, for increasing n. We can then introduce the real numbers as such `limit points' of nested interval sequences, in which we use only rational numbers as endpoints of the intervals. After this has been done, every nested interval sequence catches exactly one real number, the only point in common to all the intervals. We can take this property, the *principle of nested intervals*, as a basic property that distinguishes the real numbers from the rational numbers.

Digesting the real numbers

One of the important discoveries in the foundations of mathematics in the nineteenth century was that a tenable basis can be found for the number system by this or some similar method. Definitive constructions of the real numbers by purely logical arithmetical constructions (without appeal to intuition) were not given until the latter half of the nineteenth century, when Karl Weierstrass (1815-1897), Charles Méray (1835-1911), Richard Dedekind (1831-1916) and Georg Cantor (1845-1918), almost simultaneously and independently, each presented constructions. It should be mentioned that it was a particularly difficult problem to relate the arithmetical constructions of the real numbers to the points on a number axis.

The set of all ordered pairs of integers can be lined up in a spiral, winding its way out from $(0;0)$ in a rectangular coordinate system, and thereby an easy and visual proof can be given that this set is countable - it can be listed by the natural numbers. As a consequence, the set of rational numbers is countable. On the other hand, since you can construct a suitable nested interval sequence excluding any given countable set of real numbers from the intervals one by one, you can prove by an indirect argument, exploiting the principle of nested intervals, that the set of all real numbers is uncountable. With the chosen number axis as starting point, one can now discuss whether there are sets of real numbers of size between the set of rational numbers and the set of all real numbers in a setting where students may understand and appreciate the profound nature of the question (*the continuum hypothesis*), probably already at the secondary school level.

Using the identification of the set of real numbers with the points on a (mathematical) number axis, one can also discuss a fundamental philosophical problem in the mathematical description of the physical world. The problem has to do with the question about the existence of indivisible elements already discussed by the ancient Greek philosophers - in particular, Democritus (c. 460-400 BC). This is still an open question, and so it remains unknown whether a material physical line can be subdivided into arbitrarily small pieces. This is not a problem in the abstract mathematical world, where an interval can be subdivided *ad infinitum* in arbitrarily small subintervals. It may come as a surprise that there are already fundamental philosophical problems in the use of mathematical models in the description of the physical world at this basic level. There are good reasons to be amazed about the "unreasonable effectiveness of mathematics in the natural sciences", as Wigner puts it (see Hamming, 1980, Wigner, 1960).

On the abstraction process

In the natural sciences, it is fair to say that the abstraction process was first fully exploited by the great natural philosophers René Descartes (1596-1650), Galileo Galilei (1564-1642) and Isaac Newton (1642-1727), in connection with the formulation of the basic laws of mechanics. Maybe one can even go further and assert that the application of abstract reasoning in the quantitative study of concrete phenomena has its origin here, since qualitative abstract thinking prevailed and totally dominated quantitatively based explanations of concrete natural phenomena in the natural philosophy of Aristotle (384-322 BC) and his followers. First with Galileo the experiment became an integrated part of the scientific process in the natural sciences; (Hansen, 1993; Kline, 1985). This does not imply that the experiment became a substitute for abstract reasoning - on the contrary. Galileo's conclusion that heavy and light bodies fall to the ground with the same speed was not only based on actual experiments, but also on abstract reasoning that if a body is divided into smaller parts, then each of the parts will fall with the same speed as the body itself.

The Königsberg bridges problem

In pure mathematics a particularly good demonstration of the strength of the abstraction process can be found in connection with the solution of the Königsberg bridges problem, offered by Leonhard Euler (1707-783) in 1735. Euler's paper in this connection is widely regarded as marking the birth of the mathematical discipline of graph theory, although his solution made no mention of graphs; (Biggs, Lloyd and Wilson, 1998). We give a short description of the problem and its solution.

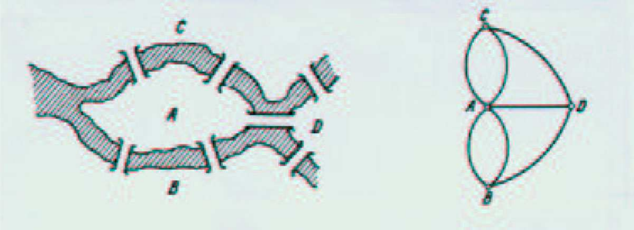

Figure 1. The Königsberg bridges problem

Where the river Pregel passes through the old Prussian town of Königsberg (now Kaliningrad) lies an island Kneipfhof at a place where the river branches into two. The four land areas separated by the river were connected by seven bridges, cf. Figure 1. The inhabitants of Königsberg had speculated whether it was possible to take a walk around town so that you cross every bridge exactly once and return to your starting point. Until Euler in 1735 found a brilliant way of tackling the problem, it remained a mystery.

The method employed by Euler was to cut away all irrelevant information (abstraction) and to represent the four pieces of land in the problem by the four letters *A*; *B*; *C*; *D* and walks along the seven bridges by words in these four letters. As an illustrative example, the word *ABADC* of length 5 represents a walk that takes you first along a bridge from *A* to *B*, then from *B* to *A*, further from *A* to *D*, and finally along a bridge from *D* to *C*. If it exists, a walk around Königsberg so that you cross every one of the seven bridges exactly once and return to your starting point must be described by a word of length 8. Since the piece of land represented by *A* is connected to the other pieces of land by five bridges, the letter *A* must occur at least 3 times in a word describing a circular tour along the seven bridges. Similarly, each of the letters *B*, *C* and *D* must occur at least 2 times, since the corresponding pieces of land are connected to the other pieces of land by three bridges. Altogether a word describing a circular tour in Königsberg as requested would therefore have to be of length at least 9, which contradicts that it should have length exactly 8. We can therefore conclude with Euler that such a circular tour in Königsberg is not possible.

A more visual solution of the Königsberg bridges problem was offered by Rouse Ball in 1892; (Rouse Ball, 1892). In this solution, the problem is modelled by a plane figure, where each piece of land is represented by a *vertex* (a point), and each bridge by an *edge* (a plane curve). Altogether the system of vertices and edges forms what is nowadays called a *graph*; see Figure 1. It should be mentioned that graphs of this kind first appeared in the late 1870s. In the graph model, the problem reduces to the question of whether you can traverse the graph in a closed cycle in such a way that you traverse every edge exactly once. Now observe that in order for this to be possible, the graph has to be connected (in one piece) and with the property that the number of edges meeting at each vertex of the graph is even (the *degree* of each vertex must be even). The particular graph associated with the Königsberg bridges problem contains vertices of odd degree and hence it is not possible to take a walk around Königsberg so that you cross every one of the seven bridges exactly once and return to your starting point.

On the nature of proof

Proofs in mathematics serve at least two important purposes: verification and explanation. From the point of view of the coherence of a mathematical theory, verification of statements is the more important. From the point of view of teaching, the explanation aspect must have a high priority. It was the Greeks who introduced proofs into mathematics; in fact, the Greek word for proof, or demonstration, also carries the meaning of explanation. In teaching, I advocate presenting proofs that have a strong component of explanation, in order to make them sensible and meaningful to the students.

Division with remainder

As a very simple example, consider the following explanation of the rule for division with remainder. Let n and m be arbitrary positive integers, where $n < m$. Choose a number axis and walk in steps of length n along the axis in the positive direction starting at 0. After a uniquely determined number of steps q, you arrive at the unique point from which the next step will bring you past m. Then, clearly, there is a unique number r (the *remainder*) in the interval $0 \leq r < n$ such that $m = nq + r$. This is the rule for division with remainder.

Using the rule for division with remainder, it is easy to prove that if, for an arbitrarily given pair of non-zero integers p; q, we consider all integers of the form $m = up + vq$, for integers u; v, and let d be the smallest strictly positive integer of this form, then d is the greatest common divisor of p and q. Accordingly, the greatest common divisor d of p and q satisfies the equation $rp + sq = d$, for some pair of integers r; s. This basic result is the starting point for a journey into elementary number theory.

The isoperimetric problem

As a more advanced example, I shall present an approach to teaching some basic elements of geometry, pointing the way to advanced mathematical theories such as optimization theory and the calculus of variations. The topic is the *isoperimetric problem* for geometrical figures in the plane bounded by closed curves. The closed curves, without self-intersections of course, can be n-gons, for each integer $n = 3, 4, \ldots$, or, more generally, rectifiable Jordan curves - curves to which a finite length can be ascribed. For each class of curves you can formulate:

The isoperimetric problem: Among all closed curves with a fixed length (in the given class), find the one that encloses the maximum area.

The formulation of the problem already raises some questions. Does there exist a solution to the problem? Is it unique? The formulation of the problem indicates that the answer to both of the questions is affirmative. This is indeed true, but it is highly non-trivial. For a discussion, see (Hansen, 1994, 1998).

Elements of the isoperimetric problem can be presented even at primary school level, and the full explanation will be accessible to mature secondary school students, see (Hansen, 1992, 1998).

First we need a result about isosceles triangles.

Theorem 4.1 Among all triangles with a given base and a given perimeter, the isosceles triangle has the largest area.

The theorem follows by observing that for all triangles with a given perimeter and a given base, the vertex opposite the base in each of these triangles lies on an ellipse with the endpoints of the base as focal points. Hence the height in such a triangle is largest at the vertex of the ellipse. This explains that the area is largest possible exactly when the triangle is isosceles over the base. We can then handle the isoperimetric problem for the class of triangles.

Theorem 4.2 Among all triangles with a prescribed perimeter, the equilateral triangle has the largest area.

The proof (explanation) of this theorem (fact) can be tackled as follows. Start with an arbitrary triangle with a prescribed perimeter. By an iterative process, we can construct a sequence of triangles with increasing area, but preserving the perimeter, by making the triangle isosceles over each one of the sides in turn. This (infinite) sequence of triangles in the limit approaches the equilateral triangle with the prescribed perimeter. By continuity of the area of a triangle on its shape, it follows that the equilateral triangle is the triangle enclosing the largest area among all triangles with the prescribed perimeter.

For the case of quadrilaterals we can solve the isoperimetric problem without involving limit processes and continuity of the area function. Surprisingly, therefore, it is easier to solve the isoperimetric problem for quadrilaterals than for triangles. The solution is as follows.

Theorem 4.3 Among all quadrilaterals with a prescribed perimeter, the square has the largest area.

The proof can be arranged as a competition among quadrilaterals with a prescribed perimeter competing for enclosing the largest area. You start out with an arbitrary quadrilateral with the given perimeter and show that you can change it into a quadrilateral with a larger area until after a few steps you end up with a square with the prescribed perimeter.

The explanation of the isoperimetric problem for quadrilaterals can be told as a short story and can easily be arranged as a play with students playing the roles of various quadrilaterals (Hansen, 1996). I know of at least one French school where this has actually been done, based on a French version of the story (Hansen, 2000). For closed curves in the plane without self-intersections and with finite length (rectifiable Jordan curves), one has the general isoperimetric problem. In this case, the circle is the solution.

Theorem 4.4 Among all closed curves in the plane without self-intersections and with a prescribed length, the circle encloses the largest area.

In (Hansen, 1994) a physical experiment is described by which you can demonstrate that the circle encloses the largest area compared to its perimeter. The experiment exhibits a closed curve which certainly appears to be a circle. But there are other figures of constant width than a circle. In fact, as discussed in (Hansen, 2002), there are figures of constant width arbitrarily close to a circle. This invites conversations on the difference between a mathematical proof and the outcome of a physical experiment, where you can be deceived by your senses.

For further reading on geometrical optimization problems related to shape and form, I recommend the wonderful book by Hildebrandt and Trompa (1996).

Aesthetics and the search for simplicity

Aesthetics holds an important place in mathematics. Somehow mathematical conjectures that are desirable in a given context and can be embodied naturally into a theory are likely to be true. When elegant and novel proofs are given for such conjectures, most mathematicians will probably agree that it is a testimony to aesthetics in mathematics.

An appealing proof

In his renowned book (Hardy, 1992), the famous number theorist G.H. Hardy includes a discussion of aesthetics in mathematics. After some thoughts he chooses to present Euclid's proof that there are infinitely many prime numbers, as an example of an aesthetically appealing proof in mathematics. Euclid's proof is one of the first proofs in mathematics that uses the method of *reductio ad absurdum*, or *proof by contradiction*. As such it is valuable in conversations with laymen about mathematical reasoning.
Theorem 5.1 There are infinitely many prime numbers.

In essence, Euclid's proof goes as follows. Suppose that 2, 3, 5, …, P is the complete list of primes. Under this hypothesis, consider the number N obtained by multiplying all these primes together, and then adding 1:

$$N = 2 \cdot 3 \cdot 5 \cdot \ldots \cdot P + 1.$$

The number N cannot be divisible by 2, for then also the difference of N and $2 \cdot 3 \cdot 5 \cdots\cdots P$ would be divisible by 2; but the difference is 1, which is not divisible by 2. In the same way, we see that N cannot be divisible by 3, or by 5, or by any of the primes up to and including P. It follows that N is a prime, or is divisible by some prime different from any of the primes 2, 3, 5,…, P. In both cases, we get a prime greater than P. This contradicts our hypothesis that 2, 3, 5,…, P was the complete list of primes, and hence this hypothesis is false.

The proof that the sequence of primes never comes to an end can easily be arranged to avoid *reductio ad absurdum*, which logicians of some schools would prefer.

From time to time there is some interest in the news media in the standing record of the largest known prime number. Due to Euclid's theorem this competition will never come to an end. On the other hand it is a famous unsolved problem in number theory as to whether there are infinitely many *pairs* of prime numbers with any given even gap between them (so-called twin primes if the fixed gap is 2).

The importance of simplicity

In mathematics it is important to strive for simple and transparent arguments, since this is the best way to ensure the correctness of results and, thus, the coherence of mathematical theories. Since coherence is vital to mathematics it is accepted as an integrated part of mathematical research to publish alternative proofs of already established original mathematical results. The goal is to find the “proof from THE BOOK”, as Paul Erdös (1913-1996) has termed brilliant proofs, which ‘feel right’ and appear to be ‘perfect’; see Aigner and Ziegler, 1998.

On philosophy of mathematics

In the preceding sections, I have touched upon several topics of a philosophical nature. It might therefore be appropriate that I explain my personal position in philosophy of mathematics in connection with a very short description of some of the main philosophies.

Probably the most important issues that a philosophy of mathematics should address are the following: (1) epistemology (how mathematical knowledge is obtained), (2) ontology (the nature of mathematical entities), (3) what is the nature of mathematical truth? (4) why can mathematics be applied to nature?

The classical philosophy of mathematics is that of Plato. Briefly, *platonists* claim that mathematical entities exist independently of our way of viewing them, and that mathematical truths are there to be discovered. In fact, Plato (427-347 BC) took the extreme view that the world of sensations is only an incomplete mirror image of the real world, which to him was the *world of ideas*, a mathematically organized world comprehensible by rational reasoning. I suppose no mathematician will go that far nowadays: rather, they would stop with Aristotle, who dismissed the existence of an independent world of ideas but discriminated between concrete and abstract aspects in the world of sensations.

A radically different philosophy of mathematics, *intuitionism*, was formulated by the Dutch mathematician L. Brouwer (1881-1966) in a famous lecture at the University of Amsterdam in 1912. It arose out of the foundational problems in mathematics discovered around 1900 in connection with the development of the theory of sets. Briefly, *intuitionists* claim that mathematical entities are mental constructs, and that mathematical statements are true only if they can be shown to be true by a constructive procedure. In particular, intuitionists reject proofs by reductio ad absurdum.

Two other main philosophies of mathematics are *formalism*, mostly associated with D. Hilbert (1862-1943), and *logicism*, mostly associated with G. Frege (1848-1925) and B. Russell (1872-1970). These also have their origins in the foundational problems in mathematics arising from set theory. *Formalists* claim that mathematical entities, if they exist at all, are nothing but terms of a formal language. *Logicists* claim that mathematical entities can be defined in the language of symbolic logic. An interesting and non-traditional comparison of the various philosophies of mathematics can be found in (Lambek, 1994).

Apart from Aristotle's version of platonism, the above philosophies of mathematics are at a rather abstract level, where the connections with concrete phenomena are weak; in fact, they are altogether independent of empirical motivation. My personal favourite for a philosophy of mathematics is to view mathematics as inspired by phenomena in nature, a kind of naturalistic platonism.

A naturalistic view on platonism

As pointed out by the zoologist D'Arcy Thompson (1860-1948) in 1917 in his remarkable book (Thompson, 1917), an overwhelming account of geometry is unfolded in the living nature around us; just think of the enchanting patterns in the wing of a butterfly, the fascinating symmetries in plants, and the fantastic shell constructions one finds among snails and mussels.

Thompson, who was a true follower of Darwin, looked for explanations of natural phenomena in terms of principles of minimization of energy and optimization of living conditions. Thompson repeats the famous quote by Galileo, who ascribes it to Plato, that “the book [of natural philosophy] is written in the mathematical language". Thompson's book certainly convinces one that there is mathematics everywhere in nature.

Following the traditions originating with Plato, it seems to me firmly established that mathematics has permanent residence in the shapes appearing in the many wonders of nature. There is a lot to be gained from this insight, for if you see the underlying mathematical structure, then you have access to the powerful universal methods of mathematics, which are tied together by the overview gained by abstracttion. In the same vein, it no longer appears a mystery that mathematics works in the description of natural phenomena, for many mathematical ideas have grown right out of nature and into our human brains.

Mathematics education and society

The public image of mathematics is that it is an important subject for the well-functioning of society and a trademark of civilization. In contemporary society this is true more than ever. The level of abstraction and complication in the mathematics applied in society has however by now reached such heights that most people, even including mathematicians, have difficulties in keeping themselves informed about recent developments. Potentially this is a dangerous situation, where society may eventually be dependent on a very few highly qualified experts. This is a main reason why mathematics education is an issue of concern to society.

It has been pointed out many times that the key to good mathematics teaching in schools is a plentiful supply of highly qualified mathematics teachers. It appears to me that in most countries the abstract parts of mathematics are neglected in teacher training. As a consequence it will then also influence the mathematics teaching in the classroom. Without proper teacher preparation, the teaching in schools will lack perspective, even though the abstract parts of mathematics should not be emphasized in the lower grades. In the selected excerpts of mathematics presented above, my intention has been to show that it might be possible to teach the abstract parts of mathematics in a concrete setting.

A comprehensive discussion of the relationship between mathematics and society can be found in (Bourguignon, 2001).

On the unity of mathematics

In the second half of the twentieth century the various mathematical subfields started to drift away from one another, and specialized conferences within still narrower subfields of mathematics became increasingly fashionable. A particularly regrettable feature of this process was that the subfields of *pure mathematics* became separated from the areas of mathematics strongly motivated by real-world problems, broadly referred to as *applied mathematics*. The same tendency was seen in the teaching of mathematics, not least at university level, where more and more specialized courses were offered - even at undergraduate level - to keep pace with the developments in mathematical research.

During this period it was nearly forgotten that mathematics occupies a very special position among the sciences and in the educational system. This position is determined by the fact that, on the one hand, mathematics is an *a priori* science building on ideal elements abstracted from sensory experiences, and on the other hand, mathematics is deeply integrated in many empirically based sciences of utmost importance for society, both in the foundations of these sciences and in the methods employed by them. Traditionally, the natural sciences and the engineering sciences strongly depend on mathematics, but nowadays also the economic and the political sciences rely on applications of mathematical methods, not least from statistics.

The mental separation between mathematicians working with abstract aspects of mathematics and those working with the more concrete aspects increased greatly during the twentieth century. The compartmentalization which followed the mental separation has led to a state of affairs not always characterized by mutual respect. As I have suggested (Hansen, 2002), the situation can and should be remedied: “The unsurpassed strength of mathematics in the description of phenomena from the outside world lies in the fascinating interplay between the concrete and the abstract. In the teaching of mathematics, and when explaining the essence of mathematics to the public, it is important to get the abstract structures in mathematics linked to concrete manifestations of mathematical relations in the outside world. Maybe the impression can then be avoided that abstraction in mathematics is falsely identified with pure mathematics and concretization in mathematics just as falsely with applied mathematics”.

By the end of the twentieth century many mathematical organizations and societies - notably, the International Mathematical Union, the American Mathematical Society and the European Mathematical Society - have realized that the strong compartmentalization in mathematics can be a threat to the vitality of mathematics in the long run. This has led to a call for the “unity of mathematics”, and many major mathematical conferences and events now include this as one of the underlying themes.

There are major reasons why the unity of mathematics is worth striving for, and the more so when it comes to mathematical education. Hence it is a pleasure to note that this issue has always been an important one for the International Congress on Mathematical Education.

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